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THE DOCTRINE

OF

LIFE-ANNUITIES AND ASSURANCES

Analytically Investigated and Practically Explained.

TOGETHER WITH

SEVERAL USEFUL TABLES CONNECTED WITH THE SUBJECT.

BY FRANCIS BAILY.

EDITED FROM THE ORIGINAL, WITH THE MODERN NOTATION, AND
ENLARGED BOTH IN THE EXTENT OF THE TREATISE,
AS WELL AS IN THE VARIETY OF TABLES.

INCLUDING

A TABLE OF DEFERRED ANNUITIES ON SINGLE LIVES, CARLISLE FOUR PER
CENT.; AND SEVERAL OTHERS ON THE ENGLISH LIFE TABLE.

BY H. FILIPOWSKI,

LATE OF THE STANDARD, THE COLONIAL LIFE OFFICES, EDINBURGH,
AND OF THE ROYAL INSURANCE OFFICE, LIVERPOOL.
AUTHOR OF A BOOK OF ANTILOGARITHMS, ETC. ETC. ETC.

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EDITOR'S PREFACE.

THE present edition of Baily's work on Life Annuities and Assurances, it is hoped, will supply a great *desideratum.* It needs no apology. Every one acquainted with the science of Life contingencies must admit that Baily's work is the best of all that has been written on the subject. It leads the learner step by step from one doctrine to another, and handles each subject with perspicuity of language, and with logical tact of an exemplary character. This work has of late years become very scarce;[1] so much so that some speculator (whose name is unknown to me), about twelve or fifteen years ago, considered it worth while to produce a *fac-simile* of the same, as regards typography, paper, &c., and sold each copy for the genuine one. Unfortunately, however, the book was edited by some one evidently ignorant of the task before him, and thus an abundance of errors and misprints was left, too numerous to be corrected with the pen, and in many cases quite impracticable to do so. Had this latter edition been revised for the press by a skilful hand, the present edition would have been almost needless. The failure of the forged copy has thus given rise to the present republication; and once decided on, it occurred to the Editor that the modern notation, as adopted by Jones, Dr. Farr, Professor De Morgan, and others, might be preferred to that used by Baily. This idea has accordingly been acted upon. The new notation is simple and intelligible at sight. A few modifications have been introduced, as will be found in the "Key to the Notation," which are so plain that a second consultation of the "Key" will scarcely ever be required.

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[1] It frequently having been sold at £4 and £5 per copy.

Several of the tables collected in Baily's original work, being all calculated on data which in the present day are seldom if ever used, such as those of *M. De Parcieux, De Moivre, The Northampton,* &c. &c., have been omitted in the present edition,[1] and in their stead others of modern date are given, calculated on data of *The Carlisle, The English,* and *The Equitable* rates of mortality. Some of these latter tables deserve particular notice. They are in fact such for the want of the like character of which the author frequently expressed regret.

Chapter XIV. of the original work, containing an account of the London offices in the Author's time, as well as the Appendix, showing a new method for calculating annuities, have likewise been omitted. The former has no direct bearing on the science of Life probabilities in general; but may be regarded as a mere criticism on the tables of the offices then in existence, while the latter is now surpassed by more simple processes, by means of the D and N columns, as explained by the Editor in §§ 37-50. It was for the same reason that the corresponding sections of the original work have been substituted by those of the Editor.

The animadversions and criticisms directed by the Author against a contemporary of his, the late Mr. Morgan, Actuary to the Equitable Society, might equally have been dropt. But on reconsideration they were suffered to remain, as they afford the student a good opportunity of investigating the several respective questions to advantage; though the style of reasoning might certainly have been less personal.

The Editor embraces this opportunity of acknowledging the liberality of the profession shown him in his publications, and begs to announce that, uniform with the present work, he is now carrying through the press the quarto book of Baily, *On Interest and Annuities*, accompanied by a large number of useful Tables for Life Insurance business; as also a collection of formulæ in great variety for all kinds of contingencies proposed and practised in our day, to be solved by means of the D and N columns. Among others, a number of auxiliary tables (such as referred to by W. T. Thomson, Esq., in his Actuarial Tables) will be given, whereby temporary and deferred annuities and assurances, on single and joint lives, may be calculated with great expedition, both on the Carlisle and the English mortality tables. The price will not exceed that of the present work.

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[1] A sufficient number of the original tables have been retained, viz., those required for the *practical questions* and examples contained in Chapter XII. In fact they form the greater part of the original tables.

H. F.

BIRKENHEAD, 1*st Feb.* 1864.

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KEY TO THE NOTATION.

l_x . . . . = The number living at age x.

l_{x+1} or l_{x1} . = The number living at age $x+1$.

l_{x-1} or l_{1x} . = The number living at age $x-1$.

$l_x l_y$ . . . = The number living at age x, multiplied by the number living at age y.

$l_x l_{y|m}$ . . = The number living at age $x+m$, multiplied by the number living at age $y+m$.

d_x . . . = The number dying at age x. All the other variations applicable to l apply also to d.

a_x . . . = The annuity on a single life aged x.

a_{x1} . . . = The annuity on a single life aged $x+1$.

a_{1x} . . . = Do. on a life aged $x-1$.

$a_{x\cdot y}$ or a_{xy} = The annuity on two joint lives aged respectively x and y.

$a_{xy|m}$ . . = The annuity on two joint lives aged respectively $x+m$ and $x-m$.

$a_{x)m}$ . . . = A *temporary* annuity for m years on a life aged x.

$a_{x(m}$ . . . = A *deferred* annuity to commence after m years on a life aged x.

r . . . . = Rate of interest.

v . . . . = $\frac{1}{1+r}$.

A . . . . = Assurance.

A_B . . . = An assurance on the life A provided he fail before B.

A_{BC} . . . = An assurance on the life A provided he fail before either B and C.

In no case are the inferior letters—when close together and unaccompanied by an algebraic character—to be regarded as factors.

AUTHOR'S PREFACE.

In the year 1808 I published a treatise on the *Doctrine of Interest and Annuities*, wherein I entered into a full investigation of all the principles relative to that science, together with its application in the various questions arising from any commercial, political, or financial inquiries. In the preface to that work I signified my intention of prosecuting the subject still further, so as to take in the whole doctrine of *Life Annuities and Assurances:* the present treatise, therefore, must be considered as a continuation of the work above alluded to, and will I believe contain all that is useful or interesting on the science. The motives which induced me to submit the *former* work to the public were there fully explained, and will equally if not more forcibly apply to the *present* treatise.

The importance of the subject at the present day cannot be doubted, since the greater part of the property of this kingdom is, in one shape or another, connected with this science. The present possessors of *entailed* estates are, in the common law, justly called *tenants for Life*, and the same appellation may be given to those who hold by *courtesy or by dower;* marriage settlements also, and wills, generally determine the possession and reversion of estates to *particular lives:* and to these contingencies every freehold estate in the kingdom is liable. If to these we add the immense number of *copyhold* estates determinable on lives, and the estates possessed by *ecclesiastical* persons of every description (all of which will probably be ever subject to the same tenure), we shall find that the value of the greater part of the *real* estates in this country will be determinable upon the principles laid down in the present work.

The incomes likewise annexed to all places, civil and military; all pensions, and most charitable donations — these, and others of a like kind, are annuities for *life*. Moreover, the dividends arising from a great part of the capital in the public funds are, by the wills of the donors and from other causes, rendered of the same nature. Besides which, many *life annuities* have been granted by individuals, by parishes, by corporate bodies, and by the Government itself. So that a great part of the *personal* estate also of this country is involved in a consideration of this subject.

In addition, however, to the cases above alluded to, there are various other circumstances in which this science will be found highly interesting and useful. There are many parents, at the present day, who are desirous of providing *Endowments* for their children, against they arrive at particular periods of life, when a sum of money is most frequently wanted—such as the time of their apprenticeship, or when they come of age, &c. Several of the Assurance Offices lately established in London, have published the rates at which they will guarantee such sums, and the present work will enable the public to determine how far it may be *prudent* to accept them.

Another interesting part of this subject is connected with the various establishments in this country, under the two general divisions, of Societies for the benefit of *Old Age*, and Societies for the benefit of *Widows*. These establishments, when founded and conducted on a true and proper basis, ought always to be encouraged, and can only be objected to when the management of their concerns is likely to fall into the hands of ignorant or designing men, who may be induced to sacrifice the permanent interest of the society to their own immediate benefit and advantage. The ruin of most of these societies may be attributed to their ignorance or neglect of the true mathematical principles upon which they ought to proceed, and without an attention to which no establishment of this kind can possibly flourish.

But the most important branch of this science is that of *Assurances*, which is still more extensive than either of those above mentioned. For, independent of the different classes of persons holding property under the several tenures alluded to in the beginning of this preface, and whose incomes will consequently determine with their lives, there is an immense number of other persons, in the different departments of society, subject to the same contingency. Every man engaged in either of the three *professions*, whose emoluments arise from his own personal abilities and exertions—every one pursuing a *naval* or *military* life, whose income will cease at his death—every person engaged in *manufactures*, *commerce*, or any other employment, whose own immediate exertions are the support of the concern in which he is engaged—these and many others, too numerous here to insist on, will often be desirous of sacrificing some part of their present emoluments and profits, not only with a view to secure a suitable provision for their families at their decease, but likewise to render their own lives more easy and comfortable, under the pleasing consolation that they have guarded against one of the great evils of a premature death.

Independent, however, of this general view of the subject, there are various other purposes for which Assurances are effected. Persons holding *Leases on Lives,* and paying a *fine* on renewal, are oftentimes induced to insure a sum of money upon those lives, in order that they may be enabled to pay such fine when it becomes due. Some consider it a good method of securing a dubious or protracted *debt*, by assuring the life of the debtor. Others, again, may be entitled to an estate, or to a sum of money, at the end of a given term, or on the happening of a particular event, provided they be then alive to receive it, and in order to secure such sum to their families, may be desirous of insuring their lives for such *term* or against such *contingency*.

These, and a thousand other cases of daily occurrence, render this branch of the science interesting to every class of the community. Numerous Offices have lately sprung up in the metropolis, for the purpose of granting Assurances on every possible contingency amongst lives in general; and it therefore becomes every one, engaged in the public business of life, to study this subject with attention.

But notwithstanding the importance and utility of these inquiries, it is not much more than a century that they have been conducted in a proper and scientific manner.[1] The celebrated Dr. Halley led the way in England; and in his paper, inserted in the *Philosophical Transactions* for 1693, pointed out the true method of calculating the value of Annuities on *Lives*. In the pursuit of this object, he assumed the rate of human mortality for five successive years, as observed at *Breslaw;* and from these data, formed the *first* correct table of the value of Life Annuities. That table, however, being adapted only to every *fifth* year of human life, and calculated at only *one* rate of interest, was consequently very limited in its application and utility.

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[1] Soon after the Revolution in this country, many of the *loans* for the service of Government were raised upon Life Annuities; and nothing can show more forcibly the low state of the science at that period, than the vague manner in which the values of such annuities were estimated.

The illustrious De Moivre improved on what Dr. Halley had begun.[1] He carefully examined the table of observations given by that celebrated philosopher: and finding that for several years together the decrements of life were uniform, and that it was only in youth and in old age that any considerable deviation occurred, he founded his ingenious hypothesis, that the decrements of life are *equal* and uniform, from birth to the utmost extremity of human life. He was at first inclined to compose a Table of the Values of Life Annuities, by keeping close to the table of observations; that is, by dividing the whole extent of human life into *several* intervals, according to the difference of the decrements during those periods. But before he undertook this task, he tried what would be the result of supposing those decrements *uniform*, from the age of twelve to the utmost extremity of life, and was satisfied, that the excesses arising on one side would be compensated by the defects on the other. For, on comparing his calculation with that of Dr. Halley, he found the conclusions to differ so very little, that he thought it superfluous to join together several different rules in order to compose a single one.

The first edition of his *Annuities on Lives* was printed in octavo in 1724. By the most simple and elegant formulæ, he pointed out the method of solving all the most common questions relative to the value of Annuities on single and joint lives, Reversions and Survivorships. In the subsequent editions of that work,[2] he not only corrected the errors

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[1] Abraham De Moivre was born at Vitri, in Champagne, in 1667. The revocation of the Edict of Nantes, in 1685, determined him with many others to take shelter in England, where he perfected his mathematical studies, the foundation of which he had laid in his own country, and which have rendered him so great an ornament to the age in which he lived. In the latter part of his life, he subsisted chiefly by giving answers to questions in Chances, Annuities, &c.; and it is said that most of these solutions were delivered at a coffee-house in St. Martin's Lane, where he spent the greatest part of his time. His merit and abilities were so well known and esteemed, that the Royal Society of London judged him a fit person to decide the famous contest between NEWTON and LEIBNITZ, concerning the invention of Fluxions. He was highly esteemed by the first of these celebrated philosophers: and it is reported, that during the last ten or twelve years of NEWTON'S life, when any person came to ask him for an explanation of any part of his works, he used to say, "Go to M. De Moivre; he knows all these things better than I do!" He died at the advanced age of eighty-seven.

[2] The *second* edition appeared in 1743, and the *third* in 1750. Since which time, I believe there have been other editions; but the most improved copy is that which is inserted at the end of his *Doctrine of Chances*, third edition, 1756.

In the preface to the second edition here alluded to, he made an illiberal and unjustifiable attack on Mr. Simpson, and charged him with mutilating his propositions, obscuring into which he had fallen in the first edition, but also greatly enlarged the boundaries of the science, and encouraged other mathematicians to pursue the path which he had struck out with so much honour to himself. Unfortunately, however, his hypothesis will not suit all circumstances; and more recent discoveries, on the rate of human mortality, have proved that it cannot always be safely adopted. Nevertheless it is still of great use in the investigation of many cases connected with this subject, and will ever remain a proof of his superior genius and abilities.

In the year 1742, Mr. Thomas Simpson published the first edition of his little treatise on the *Doctrine of Annuities and Reversions,* in which he introduced the method of computing such values from the *real observations* of life. His rules upon this subject are general, and will apply to any observations; nevertheless he confined himself, in his Table and in his Examples, to the rate of human mortality in *London,* as deduced from the observations of Mr. Smart.

The same author prosecuted this subject, by way of *Supplement,* in his *Select Exercises for Young Proficients in the Mathematics,* published in the year 1752.[1] This work, however, is for the most part a repetition of the rules given in the preceding treatise, to which are added some new problems on the subject of *contingent* annuities and assurances.

On the style of Simpson (always simple and elegant) it is needless for me to make any observations. His works are his best comment and need only be read to be admired. Nevertheless his Treatise on Life Annuities, together with his Supplement above alluded to, are perhaps the most imperfect of his productions. Although much is there done, still much more remained to be executed. His tables being deduced from the rate of mortality in *London* only, are found not to be sufficiently adapted for general use; and his Rules being deduced partly from the hypothesis

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his demonstrations, and pirating his rules. But Mr. Simpson *effectually* refuted these charges (in the same year) in an *Appendix* to his Doctrine of Annuities; at the close of which he exclaims, in the language of conscious rectitude, "I appeal to all mankind, whether, in his treatment of me, he has not discovered an air of self-sufficiency, ill-nature, and inveteracy, unbecoming a gentleman." Here the controversy appears to have dropped. For M. De Moivre published the third edition of his book without any further notice of Mr. Simpson, but omitted the offensive reflections which had been inserted in the preface to the preceding edition.

[1] That part relating to Annuities has lately been taken out of the *Select Exercises,* and having been printed separately in 1791, is now generally bound up with his *Doctrine of Annuities and Reversions* — the *second* edition of which appeared in 1775, and which contains the *Appendix* alluded to in the preceding note.

of M. De Moivre, and partly from real observations, have been ascertained not to be sufficiently correct. Subsequent improvements in the science have also shown, that some of his general theorems are erroneous; and that many cases, which frequently occur in practice, are not even mentioned in either of his works.

In 1753, Mr. James Dodson published the second volume of his *Mathematical Repository,* in which are contained not only the algebraical solutions of the problems given by M. De Moivre in his treatise above mentioned, but also several new and useful questions connected with this subject: the third volume of the same work appeared two years afterwards (1755).

In the compass of two small duodecimo volumes, the author has contrived to solve an immense variety of questions relative to Annuities, Reversions, Survivorships, and Assurances. The methods which he has pursued in investigating these cases, are in general a model of analytical reasoning, and afford an excellent *praxis* for the young mathematician. Nevertheless he is sometimes obscure, from the use of uncouth symbols, and from the culpable practice of changing the signification of his characters during the course of the same investigation. In all his solutions, he adopted the hypothesis of his friend De Moivre, conceiving that it would lead to more accurate results than the use of the Table of Life Annuities formed by Mr. Simpson (from the bills of mortality in *London*), the only one, at that time, deduced from real observations. The reader, therefore, will look in vain for any *correct* solution in the works of this author, although they may be occasionally referred to for the method of finding an *approximate* value.

The science remained in this state, without much improvement, till the publication of the first edition of Dr. Price's celebrated treatise in 1769. This work, entitled *Observations on Reversionary Payments, &c.*, was first published with a view to oppose and destroy the injurious effects and evil intentions of a class of men (unfortunately to be found in every stage of society), who, under pretence of establishing societies for the *benefit* of Old Age and of Widows, were only forming schemes to allure and to defeat the hopes of the ignorant and the distressed. His efforts were eventually crowned with success, and those *bubble* societies have long since met with the fate which he so truly predicted. In this laudable pursuit, Dr. Price saw the necessity of more accurate observations on the mortality of human life, in order to determine with more correctness the value of Life Annuities, and to show more forcibly the futility and extravagance of the schemes that were issued by those societies.[1] By the assistance of some public-spirited individuals, he obtained correct registers of the rate of mortality at Northampton, Norwich, Chester, and other places in England. But still, the computation of the value of annuities, according to these observations, was a work so tedious and unpleasant, that little hopes were entertained of profiting by those researches, and Dr. Price suffered three several editions of his treatise to pass over without affording any additional information on this subject. At length the fourth edition appeared (1783), enriched with several valuable tables of Annuities on Single and Joint Lives, at different rates of interest, deduced not only from the probabilities of living as observed at *Northampton*, but also from the probabilities of living as observed in the kingdom of *Sweden* at large.

The great addition which Dr. Price has made to our means of information respecting this science, and the assiduity with which he thus promoted some of the best interests of mankind, deserve the highest commendation: and his labours on this subject entitle him to our warmest praise. The primary object which he had in view has been fully answered; and his treatise was admirably adapted to *that* end. In every other respect, however, it is far from being complete, and the reader will look in vain for the most common cases that occur in practice. Indeed, those subjects which are to be met with do not readily present themselves, owing to the loose and irregular manner in which they are treated. Dr. Price's object was not so much to insert what was *new*, as to illustrate (by some striking examples) a few of the leading problems, with a view to oppose the pernicious schemes that disgraced the age in which he lived. But those schemes having long since vanished, his observations may now be considered rather as a *beacon* to posterity.[2]

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[1] Any person who will take the trouble to go through the Examples inserted in Dr. Price's treatise, will readily observe how inaccurately he was obliged to proceed in this infant state of the science. In calculating the value of *deferred* annuities (a case of frequent occurrence), he was obliged to take the *value of the annuity* from M. De Moivre's tables, but the *probabilities of life* he deduced from Dr. Halley's table of observations at Breslaw — a practice which gives an air of imperfection to the work at the present day, and which *ought* to have been removed after the publication of the late valuable tables.

[2] Any person the least acquainted with the subject of the present work, must be aware that any additional *Tables* of the value of Life Annuities, or any *Observations on the best method of forming them,* will add greatly to our means of information. It will therefore readily be seen, that my remarks do not allude to this part of his treatise, which I consider

The next treatise on this subject is that by Mr. Morgan, entitled *the Doctrine of Annuities and Assurances*, which appeared in 1779. This author sets out with the vain attempt to render the principles of the science intelligible to persons *unacquainted* with mathematics: but after a fruitless effort for this purpose, he ultimately leaves his readers to pursue their inquiries by the common and only useful method of *analysis*. Besides some valuable observations " on the different methods of determining the state of a society whose business consists in making Assurances on lives," that work will be found to contain a variety of problems, treated for the most part in a plain, easy, and familiar manner, and adapted to the state of the science at *that* period. But out of the forty-two problems which that treatise contains, about thirty of them, chiefly relating to *contingent* annuities and assurances, are (owing to more accurate observations and a more improved analysis) *now* rendered totally unfit for general use. Mr. Morgan himself, however, has been the principal cause of this revolution in the science; but of the *merit* of his improvements on this subject I shall speak hereafter.[1]

The last professed treatise on the science which I think worthy of notice, is Mr. Baron Maseres's *Principles of the Doctrine of Life Annuities* (1783), wherein this celebrated author has explained the subject in so familiar a manner, as to be intelligible even to those who are unacquainted with the Doctrine of Chances, and who have made no great proficiency in mathematics. This treatise, however (although consisting of more than 700 quarto pages), goes no further in the analysis of the subject, than the first two problems in the present work; but its value is greatly enhanced,

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invaluable and of constant utility. My observations, in the present instance, apply more particularly to any improvement in the *analysis* of the science, and its application to any practical cases.

[1] In Mr. Morgan's *Doctrine of Annuities*, &c., we find three *new* tables of the value of Life Annuities deduced from the probabilities of life as observed at Northampton, namely— one for single lives, another for two joint lives whose ages are equal, and another for two joint lives whose difference of age is sixty years — the interest in each table being at 4 per cent. In this infant state of the science, every additional table contributed greatly to the means of information on this subject.

It may be here necessary to remark, that the fourth edition of Dr. Price's *Observations on Reversionary Payments* (which first contained the present valuable collection of Tables) did not appear till four years *after* the publication of Mr. Morgan's work above alluded to. So that, till within these thirty years, there existed only four tables of the value of Life Annuities, viz. — two founded on M. de Moivre's hypothesis, and two deduced from the *London* observations.

by containing a variety of new Tables of the value of Annuities on Single Lives, and on two joint lives of different ages, deduced from the probabilities of living, as observed by *M. de Parcieux* amongst the Government annuitants in France—these being justly considered by the learned author as the most proper data whereon to found the value of Life Annuities. There are, moreover, in that treatise, several interesting observations on the best method of providing annuities for *Old Age*, and on various subjects of finance and political economy, which render it particularly valuable to those who are desirous of information on these important questions, and will perpetuate the name and abilities of this truly public-spirited writer.

Soon after the publication of the *fourth* edition of Dr. Price's *Observations on Reversionary Payments* (which contained the valuable collection of Tables of Life Annuities, deduced from the observations made at Northampton and in Sweden), Mr. Morgan was enabled to detect the inaccuracy of those rules which not only Mr. Simpson and others had given for determining the value of *contingent* annuities and assurances, but also which he himself had deduced from the same principles in his treatise above mentioned, and he immediately set about to correct them. His labours on this subject are contained in the several *papers* inserted by him in the *Philosophical Transactions* for 1788, 1789, 1791, 1794, and 1800. In the first volume here alluded to, he has considered those cases only in which *two* lives are concerned: in the next two volumes, his object was to deduce the value of contingent assurances in all those cases where *three* lives are concerned, and which admit of a *correct* answer: and in the last two volumes, he proposed to determine the value of contingent annuities and assurances in all the *remaining* cases of *three* lives.

Whoever will take the pains to read over those papers with attention, must be struck with surprise and regret at the *strange* and *confused* manner which Mr. Morgan has pursued, in order to obtain the solution of the several problems under consideration. No one, at the present advanced state of the science (with so many models of simplicity and elegance before him), could expect to see any mathematical inquiries conducted in so loose, so obscure, and so extraordinary a manner. The investigations are tediously and unnecessarily prolix, crowded with useless repetitions, and a variety of unmeaning quantities — all which might indeed be excused, if the resulting *formulæ* had been at once simple and correct; instead of which, we find the grossest errors committed, not only as to their *form* but as to their *accuracy*. They are for the most part *unnecessarily* long, abounding with *useless* quantities (which render their numerical solution exceedingly intricate and difficult), and oftentimes *at variance* with the particulars mentioned in the investigation, which, together with the *erroneous* manner in which they are printed, renders them of little or no use to the public. Most of his problems are investigated in *two* different ways, and are solved by the means of *two* distinct formulæ: but notwithstanding the similarity of these methods is studiously kept from the observation of the reader, and although these double formulæ are, in each problem, totally different *in appearance*, yet they will be found in *all* cases to be precisely the same, *disguised under different symbols!* A curious and interesting branch of the science has been thus strangely distorted and enveloped in mystery — a depraved taste in mathematical reasoning has been introduced — and (what is by far of the greatest importance) many *false solutions* have probably resulted from too great a dependence on the general formulæ.[1]

Mr. Morgan and myself are the only persons who have ever yet attempted to give correct solutions in the several cases of *Contingent* Annuities and Assurances. These cases have been fully investigated in the fifth and eighth chapters of the following treatise; but in conducting those investigations, I could not avoid a frequent reference to the preceding labours of Mr. Morgan on this subject, not only with a view of censuring the culpable method which he has adopted in pursuing his inquiries, but also in order to obviate any objection that might be made to *my* formulæ, *because* they do not correspond with *his*. It is needless, however, in this place, for me to add to the comments which I have already made in the two chapters above alluded to.[2]

The above are the principal English[3] authors that have written on the subject of Life Annuities and Assurances. They are few in number; and the whole of the productions, taken collectively, by no means contain a complete view of the science. And, moreover, the late improvements have rendered them, in a great measure, either obsolete or useless, and have shown the necessity of a general revision of the subject. Under these circumstances, I was induced to form a new treatise, which should comprehend not only all that is useful and important in either of the preceding works, but also such additional information as a more improved analysis and more recent discoveries in the science have been able to afford.

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[1] The *Philosophical Transactions* not being within the reach of every person, Mr. Morgan has inserted his *formulæ*, for the solution of the several problems here alluded to, in the last edition of Dr. Price's *Observations on Reversionary Payments*, note (P). But the errors of the *original* are multiplied in the copy.

[2] Several observations and notes will be found in the body of the present work, where the charges above insisted on are fully explained and demonstrated.

[3] With respect to the *foreign* writers on this science, their productions are more numerous than ours, but their inquiries are not so extensive. The subject of Life Annuities was treated by Van Hudden of Amsterdam, and likewise by the celebrated Jean de Witt, in his treatise entitled *De vardye van de lif renten, &c.* (1671). M. Struyck also inserted, in the Introduction to his *Universal Geography* (1740), some conjectures on the state of human mortality, and a long treatise on the method of calculating the value of Life Annuities.

The following is the outline of my plan—

The first chapter contains a few elementary principles of the *Laws of Chance;* some remarks on the *Probabilities of Life*, with an account of the several *Tables of Observations* made at different parts of the world;

But M. Kerseboom carried his researches much further, in his treatise published in 1748, and afterwards in 1752.

Whilst these inquiries were pursuing in Holland, M. de Parcieux was occupied with the same subject in France. In his *Essai sur la Probabilité de la Durée de la Vie Humaine* (1746), he has endeavoured to establish the rate of mortality which exists amongst life annuitants *only*, and has adopted it as a proper standard for determining the value of Life Annuities. But besides this important point, he has discussed a number of other interesting subjects connected with this science, and his work will be read with much profit and advantage. In 1779, M. St. Cyran published his *Calcul des Rentes Viageres*, which contains many useful and valuable Tables. M. de Parcieux (the nephew of the preceding author, of the same name) published also a treatise on this subject, entitled, *Traité des Annuités* (1781). But the most useful work on this science is that published by M. Duvillard, under the title of *Recherches sur les Rentes, &c.* (1787).

The researches of M. Wargentin and M. Süssmilch are well known in this country, from the frequent mention of their labours by Dr. Price, in his *Observations on Reversionary Payments.* The immortal Euler has also condescended to illustrate the first principles of this science in a paper inserted by him in the *Histoire de l'Acad. Roy. de Berlin* for 1760, wherein a method is given (similar to that of Mr. Simpson) for determining the value of an annuity on a life one year younger, from the value of an annuity on a life one year older. The same author has likewise inserted, in his *Opuscula Analytica* (1785), the solution of a question relative to Reversionary Annuities.

But notwithstanding the list of authors which is here adduced, it will be found, that as far as the *analysis* of the subject is concerned, the science remained nearly stationary under their hands. Their inquiries, in this respect, were confined principally to the method of deducing the value of annuities on single and joint lives, from given tables of observations; that is, to such subjects as are detailed in the second chapter of the present work. Those useful and interesting parts of the science which relate to the subject of Reversions, Survivorships, and Assurances, together with their several applications to the various purposes of life, do not enter into any of the foreign treatises which I have had an opportunity of seeing.

and an explanation of the general method adopted to express those probabilities in all cases. This preliminary chapter will prevent much unnecessary repetition in the course of the work.

The second chapter shows the method of determining the *Value of Annuities* on any *Single* or *Joint* Lives; on the *Longest* of any number of Lives, &c., &c. The second corollary to the first problem is of considerable importance, in enabling us to deduce, in a very easy and expeditious manner, the value of annuities on any single or joint lives from real observations. For it should be particularly observed, that tables of such values being once formed, the solutions to the subsequent problems become extremely easy, since the formulæ are expressed in terms denoting the value of such annuities.

The third chapter contains the four necessary problems for the solution of all cases of absolute *Reversionary Annuities:* and at the end of that chapter I have selected all the possible cases of two and three lives, in order that they may be more easily referred to. The formulæ there given will be found of considerable utility also, in enabling us to determine the value of the *Fines* that ought to be paid for the *Renewal of Leases* held on two or three lives, as I have fully explained in the Examples given in § 397.

The fourth chapter comprehends various cases of annuities depending on *Survivorships* between two and three lives. These cases might have been considerably augmented, but without any real benefit, since the most frequent ones are there inserted; and any other (which may arise) is easily solved by the same method of proceeding.

The fifth chapter relates to such cases of *Contingent Reversionary Annuities* as could not, for want of some previous information, be inserted in the two preceding chapters; and I believe that the method of solution, which I have there adopted, will come nearer to the correct value than any that has hitherto been published.

The sixth chapter treats of *Assurances* — a subject of great importance and extensive utility at the present day. A full explanation of the doctrine is given in the two problems inserted in that chapter.

The seventh chapter contains the method of determining the value of annuities on *successive lives;* the value of *fines* in copyhold estates held on lives; the value of *presentations, advowsons,* and things of a like kind. It likewise enables us to determine the value of the *fines* that ought to be paid for renewing or exchanging any lives held on a lease originally granted for three lives *and afterwards for a number of years certain* — a practice pursued by several corporations in this country.[1]

The eighth chapter is devoted to an investigation of the value of *Contingent Assurances*, wherein I have considered every possible case in which not more than three lives are concerned. In this branch of the science, I flatter myself that I have made considerable improvements. I have divested the subject of all extraneous matter — have not introduced more cases than were absolutely necessary — have exposed the singular formulæ given by Mr. Morgan (the only person who has preceded me in these inquiries) — and have, for the most part, introduced more correct expressions for the value of the several cases there alluded to.

The three remaining chapters complete the *analysis* of the science, and relate to such subjects as could not properly be introduced into either of the preceding ones. The ninth is confined to an explanation of the celebrated *hypothesis* of M. De Moivre, wherein its great utility and convenience, in many obvious cases, is defended against the recent attacks of Dr. Price and Mr. Morgan. The tenth treats of the method of determining the value of Life Annuities payable *half-yearly, quarterly, &c.*; also of the value of Life Annuities *secured by land*, and of the value of Assurances of sums of money payable *immediately* on the extinction of any given lives. The eleventh shows the method of finding in *Annual Payments* the value of any Assurance or of any Deferred Annuity — problems which will be found of very extensive use in practice.

The twelfth chapter contains a variety of very useful *questions* connected with this subject; to which are added the *rules* for the solution of the same, and a numerous collection of *examples*. These are thrown together into one chapter, for two obvious reasons — in the first place, by being separated from the body of the work, they do not interrupt the analytical investigations; and secondly, they may be used (together with the tables which follow) by such persons as are not acquainted with mathematics. Consequently, the present work will be accommodated to the use of both classes of readers, and (although some repetition is unavoidably occasioned thereby) may be thus rendered doubly valuable. The questions in this chapter are such as most frequently occur, but others of less public utility, or the solution to which could not be conveniently expressed in words at length, are to be met with in the body of the work, subjoined to the respective problems.

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[1] See some singular errors and absurdities into which the Corporation of *Liverpool* had fallen upon this subject, pointed out in § 412.

The thirteenth chapter shows the direct application of the sixth, thirteenth, and eighteenth questions, in the preceding chapter, to some of the most useful and important concerns of life — namely, to the method of forming the best *schemes* for providing annuities for the benefit of *Old Age* and for *Widows*. These observations are brought together under one head, in order that they might not interrupt the regular arrangement of the questions, and because it gives me, thereby, an opportunity of enlarging more fully on this very interesting subject.

Such is the nature of the present work, which will most probably terminate my labours on this subject. Much of my time is taken up in answering questions, which are laid before me for solution, relative to Annuities and Assurances. Those solutions are oftentimes different from such as arise from the ordinary rules and methods laid down by preceding writers, and it is on this account that I have been more particular in my inquiries on this subject, as well as desirous of explaining the cause of the difference, in order to remove any doubt as to their accuracy or propriety. The theorems from which my practical rules are deduced, are strictly and mathematically demonstrated in the course of the present work; and in the *numerical* enunciation of those rules (when applied to the solution of such cases as are submitted to my consideration) I discard the *indiscriminate* use of the Life Annuity Tables, deduced from the *Northampton* Observations, so generally adopted by the different Assurance Offices, and so much recommended by their immediate supporters. The motives which have influenced me to this determination it is unnecessary here to enter into, since they are fully explained in the course of the present work. And I can only add, that they will continue to be my rule of conduct as long as I am appealed to by the public as an arbiter on these subjects.

FRANCIS BAILY.

Office, No. 13, *Angel Court, Throgmorton Street,*
Feb. 12, 1810.

THE

DOCTRINE OF LIFE ANNUITIES, &c.

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CHAPTER I.

ON THE LAWS OF CHANCE AND THE PROBABILITY OF HUMAN LIFE.

§ 1. It is not my intention here to enter into a full investigation of the nature and laws of chance, but merely to explain those principles of the doctrine which are more essentially connected with the subject of the present work, in order to prevent any misunderstanding in the terms which are occasionally made use of.

§ 2. The *probability* of the happening of any event is to be understood as the ratio of the chances by which that event may happen, to all the chances by which it may either happen or fail, and it may be expressed by a fraction, whose numerator is the number of chances whereby the event may happen, and whose denominator is the number of chances whereby it may either happen or fail. Thus if there be a chances for the happening of any event, and b chances for its not happening, then will the probability of such event taking place be truly represented by $\frac{a}{a+b}$.

§ 3. In like manner, the probability of any event failing (or of its not happening) may be expressed by a fraction, whose numerator is the number of chances whereby it may fail, and whose denominator is, as before, the whole number of chances whereby it may either happen or fail. Thus the probability of the above event failing will be truly expressed by $\frac{b}{a+b}$.

§ 4. Since the sum of the two fractions representing the probabilities of the happening and of the failing of any event is equal to unity, it follows that one of them being given, the other may be found by subtraction. Thus the probability of an event happening being denoted by $\frac{a}{a+b}$, the probability of the same event failing will be truly represented by $1 - \frac{a}{a+b} = \frac{b}{a+b}$; and *vice versâ*.

§ 5. If upon the happening of an event a person be entitled to a given sum of money, his *expectation* of receiving that sum has a determinate value before the happening of the event; and such value is ascertained by multiplying the present value of the sum expected by the fraction which represents the probability of obtaining it. Thus if a person has a chances of obtaining, and b chances of losing a certain sum of money, the present value of which is equal to s, then will $s \times \frac{a}{a+b}$ denote his expectation of receiving such sum, and will be the true value of his interest therein.

Note — These principles may be more familiarly explained by the following example—Suppose that a person has three chances in five to obtain £100, the present value of his *expectation* is the product of £100 by the fraction $\frac{3}{5}$, and consequently it is worth £60. For supposing that an event may equally happen to any one of five different persons, and that the person to whom it does happen should, in consequence of it, obtain the sum of £100, it is plain that the right which each of them in particular has upon the sum expected is $\frac{1}{5}$ of £100; which right is founded on this principle, that if five persons concerned in the happening of the event should agree not to stand the chance of it, but to divide the sum expected among themselves, then each of them must have $\frac{1}{5}$ of £100 for his pretension. Now whether they agree to divide that sum equally among themselves, or rather choose to stand the chance of the event, no one has thereby any advantage or disadvantage, since they are all upon an equal footing; and consequently each person's *expectation* is worth $\frac{1}{5}$ of £100. Let us further suppose, that two of the five persons concerned in the happening of the event should be willing to resign their chance to one of the other three, then the person to whom these two chances are thus resigned has now three chances that favour him, and consequently he has now a right triple of what he had before, and therefore his expectation will in such case be worth $\frac{3}{5}$ of £100. Now if we consider that the fraction $\frac{3}{5}$ expresses the probability of obtaining the sum of £100, and that $\frac{3}{5}$ of 100 is the same as $\frac{3}{5} \times 100$, we must naturally fall into the conclusion laid down in the text, that the expectation of receiving any sum is determined by multiplying such sum by the probability of obtaining it: and though this method of reasoning is deduced from a particular case, it will easily be perceived that it is general and applicable to any other case.—See De Moivre's *Doctrine of Chances*, p. 3.

§ 6. The probability of the happening of *several events* that are independent of each other, is equal to the product of the probabilities of the happening of each event considered separately. Thus if the probability of the happening of the first of any number of independent events be denoted

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by $\frac{a}{a+b}$, that of the second by $\frac{c}{c+d}$ that of the third by $\frac{e}{e+f}$ &c., &c., then will $\frac{a}{a+b} \times \frac{c}{c+d} \times \frac{e}{e+f} \times$ &c., denote the probability of the happening of all those events. And this expression multiplied by the present value of the given sum, will denote the value of the *expectation* of receiving such sum on the happening of all those events.

Example—Suppose that in order to obtain £100, two events must happen, the first whereof has three chances to happen and two to fail, and the second whereof has four chances to happen and six to fail; the value of the expectation will in such case be $\frac{3}{5} \times \frac{4}{10} \times 100 = 24$ pounds. The demonstration of which will be very easy, if it be considered that supposing the first event had happened, the expectation (then depending entirely upon the second) would, before the determination of the second, be worth $\frac{4}{10} \times 100 = 40$ pounds. We may therefore look upon the happening of the first as a condition of obtaining an expectation worth £40; but the probability of the first event happening has been supposed $\frac{3}{5}$, wherefore the expectation sought for is to be estimated by $\frac{3}{5} \times \frac{4}{10} \times 100$; that is, by the product of the two probabilities of happening, multiplied by the sum expected. The same method of reasoning may be applied to the happening of three, or any other number of events, as may be seen more at large in the authors who have treated on this subject.

§ 7. By a similar method of reasoning, it will be evident that the probability of the failing of any number of independent events is equal to the product of the probability of the failing of each event considered separately. Thus if the probability of the failing of the first of any number of independent events be denoted by $\frac{b}{a+b}$, that of the second by $\frac{d}{c+d}$, that of the third by $\frac{f}{e+f}$, &c., &c., then will $\frac{b}{a+b} \times \frac{d}{c+d} \times \frac{f}{e+f} \times$ &c., denote the probability of the failing of all those events. And this expression, multiplied by the present value of the given sum, will denote the value of the *expectation* of obtaining such sum on the failing of all those events.

§ 8. Moreover, the probability of the happening of *either* of any number of independent events, is denoted by the difference between unity and the expression mentioned in the last article. For, since $\frac{b}{a+b} \times \frac{d}{c+d} \times \frac{f}{e+f} \times$ &c., denotes the probability that any given number of events shall fail, it follows (from § 4) that $1 - \left(\frac{b}{a+b} \times \frac{d}{e+d} \times \frac{f}{e+f} \times \text{\&c.}\right)$ will denote the probability that they shall *not* all fail, but that some one or other of them will happen. And this expression, multiplied by the present value of the given sum, will denote the value of the *expectation* of receiving such sum on the happening of either of those events.

§ 9. In like manner, if the expectation of receiving any sum depends upon the happening of any number of independent events, and upon the failing of any number of other independent events, its value will be equal to the present value of such sum, multiplied by the probability of all the former happening, and also by the probability of all the latter failing. And from these principles, we may determine the value of an expectation depending on the happening or failing of as many independent events as may be assigned.

§ 10. Hitherto I have considered only such events as are independent of each other, but if we wish to determine the probability of the happening of two events that are dependent on each other,[1] we must multiply the probability of the happening of one of them by the probability which the other will have of happening when the first is considered as having happened; and the same rule will extend to the happening of as many events as may be assigned.

§ 11. If there are several expectations upon several sums, it is evident that the expectation upon the whole will be equal to the sum of the expectations upon each. But if only one sum is to be received on the happening or failing of the given events, the method of determining the value of the expectation will be somewhat altered. The process, however, which is to be pursued in such cases, will be more fully explained in the course of the present work—what has been already said being merely introductory to the various probabilities and contingencies that occur in the following sheets.

§ 12. Now with respect to the probability that a person of a given age will or will not live to any other given age, or till a certain sum of money granted him becomes due, it is obviously in all cases a matter of very great uncertainty, and will be often very different in different persons of the same age. The chance which a man of thirty years of age, who is in good health, and lives a temperate and quiet life in the country, has to live twenty years, or till he is fifty years of age, is evidently much greater than that of another man of the same age, and of the same degree of health and vigour of body, who lives in a great city and in scenes of riot and dissipation; and it is likewise greater than that of another man of the same age, and of the same degree of health and vigour, but who is going into an unhealthy climate to which he has not been accustomed: and still more evidently, it is greater than that of another man of the same age, who is of weak and sickly constitution, or who by his daily occupation is exposed to many dangers of his life from which the generality of mankind is exempt; as is the case with soldiers and sailors, in time of war or actual service. These are circumstances beyond the reach of calculation; and all that can be done by any general rules upon this subject, is to estimate the degree of probability with which it may be reasonably expected that a person of any given age will live to any other given age, upon a supposition that he has neither a better nor a worse chance of so doing than the majority of other persons of the same age. This medium or average chance of living is determined by tables that exhibit the number of persons, which, out of a certain number of children born (usually not less than a thousand), are found by a long series of observations to be living at the end of every subsequent year of human life to its extreme period: which period in some of the tables is carried to 86, and in others to more than 90 years. The instances of the prolongation of human life to 100 years or more are so few, that they are not thought to be worth attending to in forming any general rules on this subject.

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[1] Two events are independent when they have no connexion with each other, and the happening of one neither forwards nor obstructs the happening of the other, as the continuance or failure of any given lives. On the other hand, two events may be considered as dependent, when the probability of either's happening is altered by the happening of the other; as the continuance or failure of the same life in different periods of its duration. See § 27.

§ 13. Various observations on the mortality of human life have been made by different persons and in different places; and several tables of the kind above mentioned have been calculated and formed by the different writers on this subject, such as Dr. Halley, Mr. Thomas Simpson, M. Kersseboom, M. De Parcieux, Dr. Price, M. Süsmilch, M. Wargentin, M. Muret, and others. But the same table of the probabilities of life will not suit every place; for long experience has shown, that all places are not equally healthy, or that the number of persons who die annually is different in different places. Dr. Halley formed his table from observations on the births and burials of the inhabitants of the city of *Breslaw* (the capital of the duchy of Silesia in Germany) during a series of five years, viz.—from 1687 to 1691: Mr. Thomas Simpson, from observations on the bills of mortality in *London* for ten years, from 1728 to 1737: M. Kersseboom, from the registers of certain assignable annuities for lives in *Holland*, which had been kept there for one hundred and twenty-five years, and in which the ages of the several people dying in that period had been truly entered: M. De Parcieux, from a similar use of the lists of the tontines in *France*, the numbers of which were verified by the *Nécrologes*, or mortuary registers, of several religious houses of both sexes:

Dr. Price, from a register of mortality kept at *Northampton* for forty-six years, from 1735 to 1780; the same author has also formed a table from a similar register kept at *Norwich* for thirty years, from 1740 to 1769: another from a similar register kept by Mr. Gorsuch at *Holy Cross* near Shrewsbury for thirty years, from 1751 to 1780: another from a similar register kept by Dr. Aikin at *Warrington* in Lancashire for nine years, from 1773 to 1781: another from a similar list kept by Dr. Haygarth at *Chester* for ten years, from 1772 to 1781: another from the register of mortality at *Vienna* for eight years: another from the register of mortality at *Berlin* for four years, 1752 to 1755: another from a similar register at *Brandenburgh* for fifty years, from 1710 to 1759; each of the last three being from tables given by M. Süsmilch: also another from the tables of mortality at *Stockholm* for nine years, from 1755 to 1763, as given by M. Wargentin: and another from seven different enumerations of the whole population of the kingdom of *Sweden*, each repeated at the end of three years, viz.—in 1757, 1760, 1763, 1766, 1769, 1772, and 1775. M. Muret formed his table from registers kept in forty-three parishes in the district of *Vaud* in Switzerland for ten years, from 1756 to 1765.

§ 14. All these tables differ from each other; and in many cases so materially, as to leave us in great doubt whether the subject has attained that degree of accuracy and correctness to which it is capable of being carried. It should be observed, that there are two sorts of data for forming tables of the probability of the duration of human life: one is furnished by the registers or *bills of mortality*, which show the numbers dying at all ages; the other by the proportions of deaths at all ages, to the numbers living at those ages, as discovered by surveys or *enumerations*. Those tables which are deduced from the former of these data are correct only when there is no considerable fluctuation among the inhabitants of a place, and when the births and burials are equal: for, when there are more removals *from*, than *to* a place, and the births exceed the burials (as is almost always the case in country parishes and villages), tables so formed give the probabilities of living too low: and when the contrary happens (as is generally the case in cities and large towns), they give the probabilities of living too high. But tables formed from the latter of these data are subject to no errors: they must be correct, whatever the fluctuations are in a place, and how great soever the inequalities may be between the births and the burials.

§ 15. Most of the tables above mentioned have been deduced from the former of these data; and in most of them due allowances have been made, as far as circumstances would admit, for the fluctuations arising from emigration, &c. But I believe there are no observations extant which will enable us to form tables from the latter of these data, except those published by M. Wargentin,[1] of the population of the kingdom of Sweden: and it is much to be regretted, that similar observations are not made in other countries.

§ 16. It is a singular circumstance, that not only do *females* live longer than *males*, but *married* women live longer than *single* women. All the tables of observations intimate this; but the fact has been more fully confirmed by the observations made by Dr. Aikin at *Warrington*, and by Dr. Haygarth at *Chester*, each of whom kept distinct registers of the rate of mortality amongst males and females. Similar registers also were kept at *Stockholm;* and in the enumeration of the whole population of the kingdom of *Sweden*, this circumstance was particularly attended to. These latter observations, therefore, being formed on such unerring principles, furnish sufficient data for calculating distinct tables of the value of annuities on lives among males and females, taken separately or conjunctly; and which tables might be applied with good effect in determining the value of annuities or assurances, where the lives of *widows* are concerned.[2]

§ 17. The tables of observations most used in this country at present, are those which were formed by Dr. Price from the bills of mortality at *Northampton;*[3] but they derive their importance principally from those numerous tables of the value of annuities on single and joint lives which are computed therefrom, and which afford great facility to the solution of the various cases connected with this subject. In every other point of view, it must appear extremely incorrect to take the rate of mortality in one particular town as a criterion for that of the whole country. The observations ought to be made on the kingdom at large, in the same manner as in Sweden; more particularly as, in the real business of life, the calculations are general and uniform, and adapted to persons in every situation. But till the legislature thinks proper to adopt some efficient plan for furnishing these data,[1] we must rest contented with the laudable exertions of public-spirited individuals, and avail ourselves of the best light which they afford us on this subject.

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[1] In the *Memoirs of the Academy of Sciences at Stockholm* in 1776.

[2] The circumstance here alluded to, of there being a difference in the longevity of both sexes, as also in that of married and single women, in accordance with the assertion of our author, is fully confirmed by the periodical researches, contained in the annual reports of the Registrar-General.—EDITOR.

[3] This table, except by a few of the old offices, has since been dismissed as an incorrect basis to be proceeded upon. Dr. Farr, in the Eighth Report of the Registrar-General of England, styles it "The False Table for Northampton;" inasmuch as, according to his statement, it never represented the mortality of Northampton, or of the inhabitants of All Saints' Parish, where the returns were originally made. The table of mortality at present in use in most of the life offices, is that constructed by Joshua Milne, from data collected at Carlisle by Dr. John Heysham, from 1779 to 1787 inclusive.—EDITOR.

§ 18. With respect to the several tables of mortality above mentioned, I do not think that any of them (with the exception of those given by M. Kersseboom, and M. De Parcieux) afford the proper grounds for calculating the value of *annuities*. For it is evident, that no person, in an ill state of health, or who is conscious of any thing in his constitution that might tend to the shortening of life, would give that value for an annuity which the tables indicate; neither am I inclined to think that he would become a purchaser at all. The lives, therefore, of such persons as do become annuitants, will consequently be *good* lives; or a certain part only of the general mass of mankind. The principles upon which M. Kersseboom and M. De Parcieux have formed their tables, enable us to ascertain pretty accurately the rate of mortality among this class of people, and therefore form a proper basis for determining the value of *annuities*.[2]

§ 19. The same observations may be applied, with nearly the same propriety, to the method of computing the value of *assurances*.[3] For, it is well known, that the assurer endeavours to guard as much as possible against a *bad* life: and the law of the land justly punishes any fraud in this respect. Nevertheless, as avarice or negligence may induce a relaxation of duty, the lives on which assurances are made are more liable to be *mixed* than those on which annuities are granted. In either case, however, we might often deduce a more correct value from knowing the situation of life, the residence, and mode of living of the parties concerned.

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[1] The legislature having since felt the necessity of inquiring into statistical researches, has established the office of the Registrar-General, and the results of most elaborate investigations, made by Dr. Farr, deduced from the population returns, are to be found in the annual reports of the above office. And there is no doubt, that some day, the Carlisle Table will be superseded by the English Life Table.—EDITOR.

[2] Of course these remarks of the author are allowed to remain unaltered, considering, that at the time they were written, no data excelling those alluded to by him could be obtained.—EDITOR.

[3] This fact is indubitable: for "during the last 33 years, from January 1768 to January 1801, the number of assurances on single lives [at the Equitable Society] has been 83,201; of which number sixty thousand five hundred and ninety-seven have been on the lives of persons under 50 years of age, among whom the deaths have been *fewer* than those in the Northampton table, *in the proportion of four to seven!*" (See Dr. Price's *Obs. on Rev. Pay.*, vol. ii. p. 443.) No fact can more clearly show the inaccuracy of those tables for *general* use; and though it may be *prudent* for an *insurance company* to adopt them, as well as to make use of the lowest rate of interest in calculating the values of annuities therefrom (whereby large profits are secured to the society), yet the *public*, who have no interest therein, and who occasionally seek for information on this head, should be cautious in using them, unless they appear to be applicable to the case in question. The grounds on which the calculations are made ought to be as correct as the present state of information will allow, in order that the public may be satisfied with the accuracy of the result. The contrary, however, is the fact; and Dr. Price himself has at length acknowledged it, although in rather a surreptitious manner. He introduces M. Kersseboom's and M. De Parcieux's tables of observations, in order "that nothing on this subject may be wanting;" as if it were a work of supererogation, and not one of the most essential as well as one of the most valuable parts of his treatise.

§ 20. But in many instances, both of annuities and assurances, the ages and conditions of the lives are so involved, that we must proceed upon general principles, without reference to the particular situation of the parties: and therefore, were it on this ground only, it would be extremely desirable to ascertain the rate of mortality in the kingdom at large. It would enable us to determine, how far the tables now in use might be depended upon; and furnish the basis for others more numerous and comprehensive.

§ 21. For the information of the reader I have inserted a comparative view of all the principal tables that have been given of the rate of mortality in different parts of the world; being Table I. at the end of this work. The first column shows the ages, and the other columns the number of persons living at those ages, out of 1000 born[1] at the different places mentioned at the head of each column: and these places are arranged according to their degree of mortality amongst them. London and other cities are therefore placed first; and the rest in their order, as nearly as possible, to the most healthy, which are the country provinces. This table will consequently serve to illustrate, in a striking manner, the great difference between the duration of life in large *cities* and in the *country;* for it will be seen, that in proportion as we recede from the former, the probability of life is greater, and the chance of arriving at old age is considerably increased. Thus it appears, that out of a thousand persons born at Vienna, not half of them live to be two years of age; whereas at Norwich, that number will live to be eight years of age; and at Holy Cross, they live to be above twenty-seven years of age, whilst in the province of Vaud, in Switzerland, they live to be forty-one years old. It will also fully confirm the observation which has been made in § 18, respecting the probability of living amongst those persons who purchase annuities on their own lives: for it appears from the observations of M. De Parcieux, that the chance of living amongst a set of *government annuitants* is in almost every period of their existence much *greater* than amongst an equal number of indifferent persons *living in the most healthy part of the globe;* and which consequently shows, that the Northampton tables are a very inaccurate index of the rate of mortality amongst a set of persons who purchase annuities on their own lives.

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[1] The original tables commence with numbers differing from each other; but are here reduced to the same number at the beginning, viz.—1000: by which mean we are enabled, by inspection, to compare the numbers together at any age, and immediately perceive the relative degrees of mortality at the several places given. The reader will observe that I have given other tables of the probabilities of life for *France, Sweden, Northampton,* and *London,* together with the decrements or number of persons dying annually, which being on a more enlarged scale, may be used with greater accuracy in the solution of the several problems which occur in the present work.

§ 22. But however inaccurate these tables of observations may be, or however inapplicable to existing circumstances, the subject of the present work is not at all affected thereby. For, since the principles here laid down, and the rules thence deduced, are all treated *generally*, without allusion to any particular table of observations, the reader may apply them to any of the tables above mentioned; or to any others which may be hereafter found to be more correct, or more suited to any given circumstances.

§ 23. In any table of observations, therefore, which expresses the number of persons living at every age of human life, let the number of the living at age x be denoted by l_x; and those answering to the next succeeding ages in the table by l_{x1}, l_{x2}, l_{x3}, . . . l_n, respectively; n indicating the highest age attainable according to that table. Then considering, that out of l_x persons alive at the age of x, only l_{x1} of them will be alive at the end of the year, it is evident that the number of chances for a life A_x continuing one year will be l_{x1}; and that the whole number of chances for its living or dying will be l_x; consequently, the probability that A_x will live to the end of the first year will be denoted by $\frac{l_{x1}}{l_x}$. And by a similar method of reasoning it will be seen, that the probability of his living to the end of the second year will be denoted by $\frac{l_{x2}}{l_x}$; and of his living to the end of the third year, by $\frac{l_{x3}}{l_x}$, and so on; for l_{x2}, l_{x3}, . . . l_n, will be respectively the number of chances for that life continuing 2, 3 ... $(n-x)$ years, and l_x will still be the total number of chances for his living or dying in any year.

§ 24. Moreover, the probability of any two lives, A_x and B_y, continuing in being together for 1, 2, 3, . . . n years, will be respectively denoted

$$\text{by } \frac{l_{x1}\,l_{y1}}{l_x\,l_y}, \quad \frac{l_{x2}\,l_{y2}}{l_x\,l_y}, \quad \frac{l_{x3}\,l_{y3}}{l_x\,l_y}, \ldots \frac{l_{n-x}\,l_{n-y}}{l_x\,l_y};$$

and that of three lives, A_x, B_y, C_z, by

$$\frac{l_{x1}\,l_{y1}\,l_{z1}}{l_x\,l_y\,l_z}, \quad \frac{l_{x2}\,l_{y2}\,l_{z2}}{l_x\,l_y\,l_z}, \quad \frac{l_{x3}\,l_{y3}\,l_{z3}}{l_x\,l_y\,l_z}, \ldots \frac{l_{n-x}\,l_{n-y}\,l_{n-z}}{l_x\,l_y\,l_z}.$$

§ 25. This being premised, it is evident (from § 4) that the probability of A *dying* before the end of the first year will be denoted by $1-\frac{l_{x1}}{l_x}$. For since $\frac{l_{x1}}{l_x}$ denotes the probability of his living to the end of that period, if we subtract this value from unity, it will give the probability of his not living so long. And by a similar method of reasoning, it will be found that $1-\frac{l_{x2}}{l_x}$, $1-\frac{l_{x3}}{l_x}$ . . . $1-\frac{l_{n-x}}{l_x}$, will denote the probability of the same life dying before the end of the second, third, . . . *n*th year respectively. In like manner, the probability of the life B_y failing in 1, 2, 3, . . . *n* years, will be respectively represented by $1-\frac{l_{y1}}{l_y}$, $1-\frac{l_{y2}}{l_y}$, $1-\frac{l_{y3}}{l_y}$, . . . $1-\frac{l_{n-y}}{l_y}$, and so on. Moreover, the probability that either of two lives, A, B, or of three lives, A, B, C, will fail in 1, 2, 3, . . . *n* years, will be denoted by precisely the identical expressions as in last paragraph, except that in the present case each is to be subtracted from unity, as specified above in the case of one life. And universally, if we subtract from unity the probability of the lives continuing together to the end of the given term, the remainder will express the probability that they shall not all continue together to the end of that period; but that one or other of them will die previous thereto.

§ 26. But, the probability that *all* the lives A, B, C, &c., shall fail in one year, will (by § 6) be denoted by $\left(1-\frac{l_{x1}}{l_x}\right)\left(1-\frac{l_{y1}}{l_y}\right)\left(1-\frac{l_{z1}}{l_z}\right)$;

in two years, by $\left(1-\frac{l_{x2}}{l_x}\right)\left(1-\frac{l_{y2}}{l_y}\right)\left(1-\frac{l_{z2}}{l_z}\right)$;

in three years, by $\left(1-\frac{l_{x3}}{l_x}\right)\left(1-\frac{l_{y3}}{l_y}\right)\left(1-\frac{l_{z3}}{l_z}\right)$;

&c., &c. And the probability that this event shall not happen, but that some *one* or other of the lives shall continue in being to the end of the first, second, third, &c., year, will (by § 8), be represented by

$$1-\left[\left(1-\frac{l_{x1}}{l_x}\right)\left(1-\frac{l_{y1}}{l_y}\right)\left(1-\frac{l_{z1}}{l_z}\right)\right];$$

$$1-\left[\left(1-\frac{l_{x2}}{l_x}\right)\left(1-\frac{l_{y2}}{l_y}\right)\left(1-\frac{l_{z2}}{l_z}\right)\right];$$

$$1-\left[\left(1-\frac{l_{x3}}{l_x}\right)\left(1-\frac{l_{y3}}{l_y}\right)\left(1-\frac{l_{z3}}{l_z}\right)\right].$$

§ 27. Hitherto, in deducing the probability of a life failing in any given time, I have had regard only to such event taking place *at any time before* the end of that period: but if we wish to determine the probability of the life failing *in* any particular year, the exegesis will be materially different. The probability that A will die in the second year, after having outlived the first year, is evidently equal to $1-\frac{l_{x2}}{l_{x1}}=\frac{l_{x1}-l_{x2}}{l_{x1}}$; because $\frac{l_{x2}}{l_{x1}}$ will *then* denote the probability of its living to the end of that year; and this value, being subtracted from unity, will give the probability of its *then* dying in that year: but since this event *now* depends upon its living through the preceding year (the probability of which is $\frac{l_{x1}}{l_x}$), the value above found must be multiplied by such probability, in order to give its true value: whence, the *present* value of the probability, that the life A will fail *in* the second year, is truly denoted by $\frac{l_{x1}-l_{x2}}{l_{x1}}\times\frac{l_{x1}}{l_x}=\frac{l_{x1}-l_{x2}}{l_x}$. In like manner, the probability of its failing *in* the third year, is expressed by $\frac{l_{x2}-l_{x3}}{l_{x2}}\times\frac{l_{x2}}{l_x}=\frac{l_{x2}-l_{x3}}{l_x}$: and so on to the *n*th year; when the probability of the given life A failing *in n* years will be $\frac{l_{xn}-l_{xn+1}}{l_x}$. The same observations will apply to the case of any number of joint lives; for, by pursuing the same method of reasoning, it will be found that the present probability of *either* of the three lives A, B, C, failing *in* the second year, will be denoted by $\frac{l_{x1}\,l_{y1}\,l_{z1}-l_{x2}\,l_{y2}\,l_{z2}}{l_{x1}\,l_{y1}\,l_{z1}}\times\frac{l_{x1}\,l_{y1}\,l_{z1}}{l_x\,l_y\,l_z}=\frac{l_{x1}\,l_{y1}\,l_{z1}-l_{x2}\,l_{y2}\,l_{z2}}{l_x\,l_y\,l_z}$; in the third year by $\frac{l_{x2}\,l_{y2}\,l_{z2}-l_{x3}\,l_{y3}\,l_{z3}}{l_{x2}\,l_{y2}\,l_{z2}}\times\frac{l_{x2}\,l_{y2}\,l_{z2}}{l_x\,l_y\,l_z}=\frac{l_{x2}\,l_{y2}\,l_{z2}-l_{x3}\,l_{y3}\,l_{z3}}{l_x\,l_y\,l_z}$, and so on to the *n*th year; when the probability of the lives failing *in* the *n*th year will be $\frac{l_{xn}\,l_{yn}\,l_{zn}-l_{x1+n}\,l_{y1+n}\,l_{z+n}}{l_x\,l_y\,l_z}$.

CHAPTER II.

ON LIFE ANNUITIES IN GENERAL.

§ 28. The method of determining the present value of any annuity is, to find the present value of each year's rent as it becomes due; and the sum of all these will be the total present value of the annuity required. Such value will in all cases depend on the annual rate of interest concerned; and throughout the whole of the present work I have denoted this *annual* rate[1] by r; consequently, the *amount* of £1 at the end of a year will be denoted by $(1 + r)$; and the *present value* of £1 certain to be received at the end of 1, 2, 3, &c., years, will be respectively denoted by v, v^2, v^3, v^4, &c.; the sum of which continued to n terms, or $v + v^2 + v^3 + v^4 \ldots + v^n = \frac{1 - v^n}{r}$, will denote the *present value of an annuity* of £1 per annum for n years: and if this series be continued to infinity, the sum of it, or $\frac{1}{r}$ will express the *present value of the perpetuity* of the same annuity. The principles on which these observations are founded, have been fully explained in my treatise on the *Doctrine of Interest and Annuities;* but I have thought it necessary to mention them here, in order to prevent circumlocution in the investigation of the following problems.

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[1] The *annual* rate should, in all cases of compound interest, be carefully distinguished from the nominal rate; but such annual rate may always be expressed in terms of the nominal rate, as I have distinctly shown in another work. See *Doctrines of Interest and Annuities*, p. 16.

§ 29. In life annuities, however, the rent of each year is to be received only on certain contingencies; consequently, the present values above mentioned must be diminished in proportion to the probability of receiving them: and the sum of such values, for each successive year, will be the total present value of the life annuity required. Throughout the whole of this work I have supposed the annuity to be £1 per annum; in which case, the present value deduced will denote the *number of years' purchase* that such annuity is worth; and which being multiplied by any other annuity, will give the present value of such other annuity accordingly.

PROBLEM I.

§ 30. To find the value[1] of an annuity granted for any number of lives; that is, for as long as they shall continue in being together.

SOLUTION.

Let A, B, C, &c.,[2] be the lives upon which the annuity is granted; and let the probability of each life continuing 1, 2, 3, &c., years, be as denoted in § 23; then it will follow, from what has been said in § 24, that the probability of all the lives continuing to the end of the first year will be $\frac{l_{x1}\, l_{y1}\, l_{z1}}{l_x\, l_y\, l_z}$: which, being multiplied by v^1, or the present value of £1 certain to be received at the end of one year, will produce $v\frac{l_{x1}\, l_{y1}\, l_{z1}}{l_x\, l_y\, l_z}$ for the present value of the first year's rent; or the *expectation* of receiving such sum on the contingency that all the lives continue to the end of the first year. In like manner, since the probability that all the lives will continue to the end of the second year is $\frac{l_{x2}\, l_{y2}\, l_{z2}}{l_x\, l_y\, l_z}$, if this expression be multiplied by v^2, or the present value of £1 certain to be received at the end of two years, it will produce $v^2\frac{l_{x2}\, l_{y2}\, l_{z2}}{l_x\, l_y\, l_z}$ for the present value of the second year's rent; or the *expectation* of receiving such sum on the contingency that all the lives will continue to the end of the second year. By the same method of reasoning, it will be found that $v^3\frac{l_{x3}\, l_{y3}\, l_{z3}}{l_x\, l_y\, l_z}$ will denote the present value of the third year's rent; or the *expectation* of receiving such sum on the contingency that all the lives will continue to the end of the third year. And in this manner we must proceed for all the subsequent years of human life, the sum of all which terms,[1] or $\frac{1}{l_x\, l_y\, l_z}(v\, l_{x1}\, l_{y1}\, l_{z1} + v^2\, l_{x2}\, l_{y2}\, l_{z2} + v^3\, l_{x3}\, l_{y3}\, l_{z3} + \ldots v^n\, l_{x+n}\, l_{y+n}\, l_{z+n})$ will be the total present value of the annuity; where *n* in this case denotes the number of years between the age of the oldest of the given lives, and the age of the oldest life in the table of observations.

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[1] This fundamental proposition, upon which the whole doctrine of annuities in a great measure depends, may be found in most authors who have treated on this subject. In the investigation of the subsequent Problems and their Corollaries, I shall refer to the similar propositions in the works of the five following authors, viz.— Simpson's *Doctrine of Annuities and Reversions,* 1775, and his *Supplement* to the same, 1791; De Moivre's *Doctrine of Chances,* 3rd edition, 1756; Dodson's *Mathematical Repository;* Dr. Price's *Observations on Reversionary Payments,* 6th edition, 1803; Morgan's *Doctrine of Annuities and Assurances, Philosophical Transactions* for 1788, 1789, 1791, 1794, and 1800; whereby the reader may the more readily compare them together, and judge of the respective merits of the rules which they have given for the solution of the same.

By the term *value,* I mean the number of years' purchase that the annuity is worth, agreeable to what I have just observed. This mode of expression will be used throughout the present work.

[2] For the sake of brevity, and for the typographical appearance, I will henceforth, in

COROLLARY I.

§ 31. Now, when only one life A is concerned, this series will be

$$\frac{1}{l_x}(v.l_{x1} + v^2\, l_{x2} + v^3\, l_{x3} + \ldots v^n\, l_{x+n});$$

when two lives, A and B, are concerned, it will become

$$\frac{1}{l_x\, l_y}(v.l_{x1}\, l_{y1} + v^2\, l_{x2}\, l_{y2} + v^3\, l_{x3}\, l_{y3} + \ldots v^n\, l_{x+n}\, l_{y+n}):$$

Hence, if we make a_x to denote *the value of an annuity* on any single life aged x; and $a_{x \cdot y}$ to denote *the value of an annuity* on any two joint

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most cases, dismiss the appendage "&c.," too fondly carried along by the author in almost each term; it being understood, that from the formulæ for two or three lives, others may be formed also for four or any number of lives. — EDITOR.

[1] When the sum of this series is to be determined in numbers, the terms of it must be carried to the extinction of the oldest life C, involved in the combination; at which period all the subsequent terms vanish, because l_{z1}, l_{z2}, become equal to nothing. This observation will apply to all cases of combined lives.

lives, aged respectively x and y; and $a_{x\cdot y\cdot z}$ to denote *the value of an annuity* on any three joint lives aged respectively x, y, and z; then, in the case of a single life, we shall have

$$a_x = \frac{1}{l_x}(v\,.\,l_{x1} + v^2\,l_{x2} + v^3\,l_{x3} \ldots + v^n\,l_{x+n});$$

and in the case of two joint lives, we shall have

$$a_{xy} = \frac{1}{l_x\,l_y}(v\,.\,l_{x1}\,l_{y1} + v^2\,l_{x2}\,l_{y2} + v^3\,l_{x3}\,l_{y3} \ldots v^n\,l_{x+n}\,l_{y+n});$$

and in case of three joint lives, we shall have

$$a_{x\cdot y\cdot z} = \frac{1}{l_x\,l_y\,l_z}(v\,.\,l_{x1}\,l_{y1}\,l_{z1} + v^2\,l_{x2}\,l_{y2}\,l_{z2} + v^3\,l_{x3}\,l_{y3}\,l_{z3} \ldots v^n\,l_{x+n}\,l_{y+n}\,l_{z+n}).$$

COROLLARY II.

§ 32. Although there is no method for summing up these terms, or for abridging the general expression above given for finding the value of annuities on single and joint lives, but each respective term must be actually reduced to numbers, and the whole of them added together in order to determine the total value of the annuity; yet in finding the value of annuities on a number of single or joint lives, that is on lives of several successive ages, the process may be considerably abridged by deducing the value of an annuity on the next younger life from the value of an annuity on a life or lives each one year older. For, let $a_{x\cdot y\cdot z}$ denote the value of an annuity on any number of joint lives A, B, C; and $a_{x-1\cdot y-1\cdot z-1}$ the value of an annuity on the same number of joint lives each one year younger than A, B, C; and let $l_{x-1}\,l_{y-1}\,l_{z-1}$ be the number of persons living at the ages of those younger lives, as found in any table of observations: then, for the very same reason that

$$\frac{1}{l_x\,l_y\,l_z}(v\,.\,l_{x1}\,l_{y1}\,l_{z1} + v^2\,l_{x2}\,l_{y2}\,l_{z2} + v^3\,l_{x3}\,l_{y3}\,l_{z3} \ldots + v^n\,l_{x+n}\,l_{y+n}\,l_{z+n}) = a_{x\cdot y\cdot z},$$

we shall have

$$\frac{1}{l_{x-1}\,l_{y-1}\,l_{z-1}}(v\,.\,l_x\,l_y\,l_z + v^2\,l_{x1}\,l_{y1}\,l_{z1} + v^3\,l_{x2}\,l_{y2}\,l_{z2} \ldots v^{n-1}\,l_{x+n-1}\,l_{y+n-1}\,l_{z+n-1})$$
$$= a_{x-1\cdot y-1\cdot z-1}.$$

Wherefore, multiplying the first equation by $l_x\,l_y\,l_z$, and the latter by $\frac{l_{x-1}\,l_{y-1}\,l_{z-1}}{v}$ we have

$$v\,.\,l_{x1}\,l_{y1}\,l_{z1} + v^2\,l_{x2}\,l_{y2}\,l_{z2} \ldots = a_{x\cdot y\cdot z} \times l_x\,l_y\,l_z;$$

$$v\,.\,l_{x1}\,l_{y1}\,l_{z1} + v^2\,l_{x2}\,l_{y2}\,l_{z2} \ldots = a_{x-1\cdot y-1\cdot z-1}\left(\frac{l_{x-1}\,l_{y-1}\,l_{z-1}}{v}\right):$$

whence $a_{x-1\cdot y-1\cdot z-1}\,\frac{l_{x-1}\,l_{y-1}\,l_{z-1}}{v} - l_x\,l_y\,l_z = a_{x\cdot y\cdot z} \times l_x\,l_y\,l_z$.

Consequently $a_{x-1\cdot y-1\cdot z-1} = (1 + a_{x\cdot y\cdot z}) \times \frac{v\,.\,l_x\,l_y\,l_z}{l_{x-1}\,l_{y-1}\,l_{z-1}}$;

whence the following rule for finding the value of an annuity on any *single* life, the principle of which it is easy to apply to the case of any *joint* lives. Begin with the oldest life in the table of observations; add unity to the value of an annuity on that life (usually equal to 0) and multiply the sum by the expectation of *a life one year younger* receiving £1 at the end of a year; the product will be the value of an annuity on the life one year younger: this value being substituted for the value of an annuity on the oldest life, and the process repeated, will give the value of an annuity on the next youngest life: and so on till we come to the age of the given life. Now, though the method of deducing the value of an annuity on any *given* life by means of this formula is rather more laborious than finding the numerical value of each term of the series given in the last corollary; yet the present formula has this advantage, that the several steps of the process give the values of annuities on lives of all the ages between the given life and the oldest life in the table of observations: whence the calculation of the value of an annuity on lives of *all* those ages becomes scarcely more troublesome than the calculation of the value of an annuity on the youngest life.

§ 33. Example 1—Let it be required to find the value of an annuity on a life aged 95 years, allowing interest at the rate of 3 per cent. per annum, and according to the probabilities of life in general as observed (among males and females collectively) in Carlisle.

The value of an annuity on a life aged 104 is evidently equal to 0:

$$\begin{aligned}
a_{103\cdot} &= (1 + 0{\cdot}00000) \times \tfrac{1}{3} \times {\cdot}97087 = {\cdot}32362\\
a_{102\cdot} &= (1 + 0{\cdot}32362) \times \tfrac{3}{5} \times {\cdot}97087 = {\cdot}77104\\
a_{101\cdot} &= (1 + 0{\cdot}77104) \times \tfrac{5}{7} \times {\cdot}97087 = 1{\cdot}22819\\
a_{100\cdot} &= (1 + 1{\cdot}22819) \times \tfrac{7}{9} \times {\cdot}97087 = 1{\cdot}68256\\
a_{99\cdot} &= (1 + 1{\cdot}68256) \times \tfrac{9}{11} \times {\cdot}97087 = 2{\cdot}13089\\
a_{98\cdot} &= (1 + 2{\cdot}13089) \times \tfrac{11}{14} \times {\cdot}97087 = 2{\cdot}38834\\
a_{97\cdot} &= (1 + 2{\cdot}38834) \times \tfrac{14}{18} \times {\cdot}97087 = 2{\cdot}55861\\
a_{96\cdot} &= (1 + 2{\cdot}55861) \times \tfrac{18}{23} \times {\cdot}97087 = 2{\cdot}70389\\
a_{95\cdot} &= (1 + 2{\cdot}70389) \times \tfrac{23}{30} \times {\cdot}97087 = 2{\cdot}75694
\end{aligned}$$

whence it appears that the value required is equal to 2·75694.[1]

§ 34. Example 2—What is the value of an annuity on the joint lives of a man aged 97, and a woman aged 91, reckoning interest at 3 per cent., and the probabilities of living, among males and females *collectively*, as observed in *Carlisle*.

By beginning with the oldest life in the table of observations, joined to another whose difference of age is 6 years, it will be found that the value of an annuity on two similar joint lives, aged 104 and 98, is equal to 0:

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[1] Dr. Price has given *incorrect* values for annuities on *lives* in *general*, according to the table of observations for Sweden. See *Obs. on Rev. Pay.* vol. ii. p. 422. He has taken a *mean* between the value of annuities on male and female lives, which is evidently erroneous.

$$
\begin{aligned}
a_{103\cdot97} &= (1 + 0{\cdot}000) \times \tfrac{1}{3} \times \tfrac{14}{18} \times {\cdot}97087 = 0{\cdot}252\\
a_{102\cdot96} &= (1 + 0{\cdot}252) \times \tfrac{3}{5} \times \tfrac{18}{23} \times {\cdot}97087 = 0{\cdot}571\\
a_{101\cdot95} &= (1 + 0{\cdot}571) \times \tfrac{5}{7} \times \tfrac{23}{30} \times {\cdot}97087 = 0{\cdot}835\\
a_{100\cdot94} &= (1 + 0{\cdot}835) \times \tfrac{7}{9} \times \tfrac{30}{40} \times {\cdot}97087 = 1{\cdot}039\\
a_{99\cdot93} &= (1 + 1{\cdot}039) \times \tfrac{9}{11} \times \tfrac{40}{54} \times {\cdot}97087 = 1{\cdot}200\\
a_{98\cdot92} &= (1 + 1{\cdot}200) \times \tfrac{11}{14} \times \tfrac{54}{75} \times {\cdot}97087 = 1{\cdot}208\\
a_{97\cdot91} &= (1 + 1{\cdot}208) \times \tfrac{14}{18} \times \tfrac{75}{105} \times {\cdot}97087 = 1{\cdot}191
\end{aligned}
$$

whence the value of the annuity required is equal to 1·191.

§ 35. But the best mode of computing the values of annuities is by means of *logarithms:* and the method of doing this will be sufficiently evident from an inspection of the formula in page 29, and from the principles which have been just laid down. For, since $a_{x\cdot y\cdot z\cdot} = (1 + a_{x1\cdot y1\cdot z1}) \times \frac{l_{x1}\, l_{y1}\, l_{z1}}{l_x\, l_y\, l_z}\, v$, it is manifest that $\log.\ a_{x\cdot y\cdot z\cdot} = \log.\ (1 + a_{x1\cdot y1\cdot z1\cdot}) + \log.\ l_{x1} + \log.\ l_{y1} + \log.\ l_{z1} + \log.\ v - [\log.\ l_x + \log.\ l_y + \log.\ l_z]$: whence the calculations, which appear so laborious and intricate when 2 or 3 joint lives are involved, are reduced to the simple operations of addition and subtraction.

§ 36. In calculating the values of annuities according to this method, the following directions should be observed. Begin with the oldest life, C, and write down horizontally on a paper (divided into columns, as in the annexed specimen) the logarithms of the number of persons living at the end of n, $(n-1)$, $(n-2)$, &c. years from birth; n denoting the number of years between *birth* and the age of the oldest life in the table of observations. Then, proceeding to the next oldest life B, write down in a similar manner, under the former, the logarithms of the number of persons living at the end of $(n-d)$, $(n-d-1)$, $(n-d-2)$, &c. years from birth: d denoting the difference of age between B and C. And so on to the next oldest life, according to the number of joint lives required. Add these several perpendicular values together, and write down under each of them the logarithm of v. This being done, the subsequent operations become extremely easy, as will sufficiently appear from the following specimen, which shows the method of obtaining the values of annuities on two joint lives of all ages whose difference of age is 10 years; reckoning interest at 4 per cent., and the probabilities of living as at *Northampton.*

Note.—Since the specimen here alluded to by the Author is now superseded by more improved processes, as will be explained in the sequel, it is omitted in this edition.—EDITOR.

§ 37. Having shown above the method pursued by the author for calculating annuities on single and joint lives, I now proceed to illustrate the process employed by Griffith Davies. Seeing that

$$a_x = \frac{vl_{x1} + v^2 l_{x2} + v^3 l_{x3} \ . \ . \ . + v^n l_{xn}}{l_x} \quad (1.)$$

if therefore the numerator and denominator of this fraction be multiplied by v^x (which will not affect the value of the expression), the formula becomes

$$a_x = \frac{v^{x+1} l_{x1} + v^{x+2} l_{x2} + v^{x+3} l_{x3} \ . \ . \ . + v^{x+n} l_{xn}}{v^x l_x} \quad (2.)$$

Thus we obtain the following rule: *Multiply the number of living at each year of age by the present value of £1, due at the end of the same number of years as the age given; then the present value of the annuity at any age is found, by dividing the sum of the products at all ages above that on which the annuity depends, by the product at that age.*

§ 38. The advantage of the second formula over the other may be seen by taking as examples the separate ages of 96 and 95, in the Carlisle table of mortality.

$$a_{96} = \frac{v^{97} l_{97} + v^{98} l_{98} + v^{99} l_{99} \ . \ . \ . + v^{104} l_{104}}{v^{96} l_{96}}$$

$$a_{95} = \frac{v^{96} l_{96} + v^{97} l_{97} + v^{98} l_{98} \ . \ . \ . + v^{104} l_{104}}{v^{95} l_{95}} .$$

On comparing the expressions for these two values, we observe, that in finding the value for age 95, every term is introduced which was employed in finding the value for age 96; so that it is not more troublesome to find the value for both ages than to find the value for one of them only. But had the first formula been used, the operation employed in finding the value at the age of 96 would not have afforded direct assistance in finding the value at the age of 95. This second formula has also other important advantages, the preparatory operations being of great service in abridging the labour of finding the values of temporary and deferred annuities and assurances.

The following example, in numbers, of the value of annuities at 3 per cent., by the Carlisle rate of mortality, will show the process of forming a table of annuities on single lives.[1]

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[1] §§ 37 and 38 slightly modified, I have extracted from the work of David Jones on *Annuities*, pp. 114, 115.—EDITOR.

$$l_{104}\,v^{104} = 1\times\cdot0462305 = N_{104} = \cdot0462305 \qquad a_{103} = \frac{\cdot0462305}{\cdot1428522} = 0\cdot32362$$
$$l_{103}\,v^{103} = 3\times\cdot0476174 = D_{103} = \cdot1428522$$
$$N_{103} = 1890827 \qquad a_{102} = \frac{\cdot1890827}{\cdot2452295} = 0.77104$$
$$l_{102}\,v^{102} = 5\times\cdot0490459 = D_{102} = \cdot2452295$$
$$N_{102} = \cdot4343122 \qquad a_{101} = \frac{\cdot4343122}{\cdot3536211} = 1\cdot22819$$
$$l_{101}\,v^{101} = 7\times\cdot0505173 = D_{101} = 3536211$$
$$N_{101} = \cdot7879333 \qquad a_{100} = \frac{\cdot7879333}{\cdot4682952} = 1\cdot68256$$
$$l_{100}\,v^{100} = 9\times\cdot0520328 = D_{100} = \cdot4682952$$
$$N_{100} = 1\cdot2562285 \qquad a_{99} = \frac{1\cdot2562285}{\cdot5895318} = 2\cdot13089$$
$$l_{99}\,v^{99} = 11\times\cdot0535938 = D_{99} = \cdot5895318$$
$$N_{99} = 1\cdot8457603 \qquad a_{98} = \frac{1\cdot8457603}{\cdot7728224} = 2\cdot38834$$
$$l_{98}\,v^{98} = 14\times\cdot0552016 = D_{98} = \cdot7728224$$
$$N_{98} = 2\cdot6185827 \qquad a_{97} = \frac{2\cdot6185827}{1\cdot0234386} = 2\cdot55861$$
$$l_{97}\,v^{97} = 18\times\cdot0568577 = D_{97} = 1\cdot0234386$$
$$N_{97} = 3\cdot6420213 \qquad a_{96} = \frac{3\cdot6420213}{1\cdot3469582} = 2\cdot70389$$
$$l_{96}\,v^{96} = 23\times\cdot0585634 = D_{96} = 1\cdot3469582$$
$$N_{96} = 4\cdot9889795 \qquad a_{95} = \frac{4\cdot9889795}{1\cdot8096090} = 2\cdot75694$$
$$l_{95}\,v^{95} = 30\times\cdot0603203 = D_{95} = 1\cdot8096090$$

§ 39. In like manner, a table of annuities may be prepared for two, three, or more joint lives: by introducing always at each step, the number living at the age of the other life or lives. Suppose a table of annuities, at the Carlisle rate of mortality and three per cent. interest, is to be computed for two lives differing in their ages with five years, the calculation in numbers will be as follows—

$$l_{104}\,l_{99}\,v^{104} = 1\times11\times\cdot0462305 = N_{104\cdot99} = \cdot5085355 \qquad a_{103\cdot98} = \frac{\cdot5085355}{1\cdot9999308} = 0\cdot25428.$$
$$l_{103}\,l_{98}\,v^{103} = 3\times14\times\cdot0476174 = D_{103\cdot98} = 1\cdot9999308$$
$$N_{103\cdot98} = 2\cdot5084663 \qquad a_{102\cdot97} = \frac{2\cdot5084663}{4\cdot4141310} = 0\cdot56828.$$
$$l_{102}\,l_{97}\,v^{102} = 5\times18\times\cdot0490459 = D_{102\cdot97} = 4\cdot4141310$$
$$N_{102\cdot97} = 6\cdot9225973 \qquad a_{101\cdot96} = \frac{6\cdot9225973}{8\cdot1332884} = 0\cdot85114.$$
$$l_{101}\,l_{96}\,v^{101} = 7\times23\times\cdot0505173 = D_{101\cdot96} = 8\cdot1332884$$
$$N_{101\cdot96} = 15\cdot0558857 \qquad a_{100\cdot95} = \frac{15\cdot0558857}{14\cdot0488668} = 1\cdot07168.$$
$$l_{100}\,l_{95}\,v^{100} = 9\times30\times\cdot0520328 = D_{100\cdot95} = 14\cdot0488668$$
$$N_{100\cdot95} = 29\cdot1047525$$

In short, the column D_x is always the product obtainable from the number living at age x being multiplied by the present value of £1 due at the end of x years; whilst the column N_x is formed from the summations of column D; always taking its commencement from the highest age of the table of mortality, and proceeding downwards to age x inclusive. Or, in the case of two lives aged respectively x y (supposing x to be the age of the older life), D_x is the product obtained from the numbers living at x and at y being multiplied by the present value of £1 due at the end of x years. Thus, D and N for the highest age, that is, the first values with which the computor is to start, are always one and the same. But each successive value of N may generally be regarded as the total of the series from the highest age down to the age given, and so forms the result of the formula in numbers. Hence each rate of mortality, combined with any one rate of interest, requires its special D and N columns. Mr. Davies, in his D and N columns, preferred to shift the latter one line downwards, so that his N_x always corresponds with D_{x-1}, or N_{x+1} with D_x. No reason can be assigned for this arbitrary change, unless it be, that he wished the expression for finding an annuity to be always $\frac{N_x}{D_x}$, and not, as it properly ought to be $\frac{N_{x+1}}{D_x}$. — EDITOR.

§ 40. It will be evident to the learner, that the method of the D and N columns is a decided improvement upon that adopted by our author. For, by means of the latter, an annuity on a life aged x can only be obtained by the aid of all the successive values of annuities, from the highest age in the table down to the age x; whilst by the D and N columns, such annuity can be found by a single operation, viz., by dividing N_{x+1} by D_x. But it might apparently be questioned, whether the amount of labour required for the construction of the D and N columns, in addition to the required division of each N by each corresponding D, does not actually exceed that necessary for the process indicated by our author? This query may be answered in the affirmative, so far as whole-life annuities only are concerned; but when we come to consider that the same D and N columns afford the means of finding with facility the values of temporary and deferred annuities or assurances, as also the values of others equally important, then the extra labour originally incurred must be entirely disregarded, as will be more clearly demonstrated presently. — EDITOR.

COROLLARY III.[1]

§ 41. By means of the general expression in the problem, for finding the value of an annuity on any given lives, we may determine the value of a *Deferred* life annuity; that is, of an annuity which is not to commence till the end of a given number of years ($=m$), provided the given lives are then in being, and to continue from that period to the extinction of such given lives. For, let A, B, C, be the given lives; and let l_{x+m}, l_{y+m}, l_{z+m}, denote the number of persons living, according to any table of observations, at the end of m years, from the several ages of A_x, B_y, C_z, respectively, as explained in § 23; then the value of a deferred annuity on the lives A B C, to be entered on,[1] at the end of m years will be expressed by the following series —

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[1] Price, Note (B). Simpson's Sup. Prob. 13.

$$\frac{1}{l_x\, l_y\, l_z}\,(v \cdot l_{x1+m}\, l_{y1+m}\, l_{z1+m} + v^2 l_{x2+m}\, l_{y2+m}\, l_{z2+m} + v^3\, l_{x3+m}\, l_{y3+m}\, l_{z3+m} \ldots + v^{n-x} l_n);$$

the sum of which I shall denote by $a_{x\cdot y\cdot z(m}$. Now, if we have tables of the values of annuities on single and joint lives of all ages, we may easily deduce the value of this series (without the actual calculation of each separate term) *from the value of an annuity on the same number of lives each* m *years older than the given lives.* For, if this latter value be denoted by $a_{x\cdot y\cdot z|m}$, we shall have, according to the principles laid down in the problem —

$$a_{x\cdot y\cdot z|m} = \frac{1}{l_{x+m}\;\, l_{y+m}\;\, l_{z+m}}\,(v \cdot l_{x1+m}\;\, l_{y1+m}\;\, l_{z1+m} + v^2\, l_{x2+m}\;\, l_{y2+m}\;\, l_{z2+m} + \;.\;.\;.\;);$$

therefore, multiplying both sides of this equation by $\dfrac{v^m l_{x+m}\;\, l_{y+m}\;\, l_{z+m}}{l_x\, l_y\, l_z}$, we produce $a_{x\cdot y\cdot z|m} \times \dfrac{v^m l_{x+m}\;\, l_{y+m}\;\, l_{z+m}}{l_x\, l_y\, l_z} = a_{x\cdot y\cdot z(m}$: whence the following general rule.

§ 42. *Find the value of an annuity on the same number of lives, each as many years older than the given lives as are equal to the number of years during which the annuity is deferred; find also the expectation*[2] *of the given lives receiving £1 at the end of that term: the product of these two quantities will be the value required.*

For examples of the use and application of this corollary, see Question VI. in Chapter XII.

COROLLARY IV.[3]

§ 43. By the help of the series in the last corollary we may deter-

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[1] The first payment of which, however, will not take place till the end of the $(n+1)^{st}$ year. See my *Doctrine of Interest and Annuities*, p. 65, note.

[2] The *expectation* of receiving any sum of money at the end of any given term, is equal to the present value of such sum multiplied by the probability that the given lives will continue in being so long: that is, equal to

$$\frac{v^n\, l_{x+m}\, l_{y+m}\, l_{z+m}}{l_x\, l_y\, l_z}.$$ See chap. i. § 5 and 6.

[3] Simpson's Sup. Prob. 6. De Moivre, Prob. 26. Morgan, Prob. 4. Dodson, vol. iii. Ques. 2, 7, and 10.

mine the value of a *Temporary* life annuity: that is, of an annuity which is to commence immediately, but to continue only for a given number of years ($=m$) which is less than that to which it is possible that the given life or lives may extend,[1] and then to cease. For, supposing every thing to remain as in the last corollary, it is evident, that the first m terms[2] of the series there given, or $\frac{1}{l_x\,l_y\,l_z}(v.\,l_{x1}\,l_{y1}\,l_{z1}+v^2\,l_{x2}\,l_{y2}\,l_{z2}+v^3\,l_{x3}\,l_{y3}\,l_{z3}\ldots v^m l_m)$, will be the value of the temporary annuity required, and which I shall denote by $a_{x\cdot y\cdot z)m}$. Now by means of the tables above mentioned, we may easily deduce the value of this series without the actual enumeration of its several terms: for, it is evident on inspection that the value of the first m terms of the series in the last corollary is equal to the value of the whole series *minus* the sum of all the terms after the m^{th} term. But the sum of all the terms after the m^{th} term has been found, by the last corollary, to be equal to $a_{x\cdot y\cdot z(m}$: consequently, the sum of the first m terms is equal to $a_{x\cdot y\cdot z}-a_{x\cdot y\cdot z(m}$; whence the following rule —

§ 44. *Find the value of an annuity on the given lives deferred during the given period, and deduct this value from the value of an annuity on the given lives: the difference will be the value required.*

For examples of the use and application of this corollary, see Question VII. in Chapter XII.

COROLLARY V.

§ 45. By a similar method of reasoning we may find the value of a *Deferred and Temporary* life annuity; that is, of an annuity which is not to commence till the end of a given number of years ($=m$), and then to continue only during another period ($=n$), which is less than that to which it is possible the given lives may *then* be prolonged. And, from what has been said in the third corollary, the truth of the following rule will be evident.

§ 46. *From the value of an annuity on the given lives, deferred for the term* m, *subtract the value of the same annuity on such lives, deferred for the term* $(m+n)$; *the difference will be the sum of the required* n *terms of the series, or the value of the annuity proposed.*

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[1] This is always understood in cases of this kind: and it may be here useful to remark, that the term during which it is possible any given life may be prolonged, is equal to the difference between the age of that life, and the age of the oldest life in the table of observations. In the case of joint lives, it is equal to the difference between the oldest of such joint lives, and the age of the oldest life in the table of observations.

[2] By the *first m terms*, the reader is to understand the primitive terms without the appendage m. — EDITOR.

§ 47. By the D and N columns specified above, temporary, deferred, and intercepted annuities can be obtained with equal facility, as in the case of whole-life annuities. It is evident (from § 23) that the probability of a life aged x living *one* year is $\frac{l_{x1}}{l_x}$, *two* years $=\frac{l_{x2}}{l_x}$, and m years $=\frac{l_{x+m}}{l_x}$. It will likewise be observed (from § 28), that the present value of £1 to be received at the end of *one* year is v; at the end of *two* years $= v^2$, and at the end of m years $= v^m$; consequently (as demonstrated in § 31), the probability of a life aged x receiving £1 at the end of *one* year is $\frac{l_{x1}v}{l_x}$, at the end of *two* years $=\frac{l_{x2}v^2}{l_x}$, and at the end of m years $\frac{l_{xm}v^m}{l_x}$; hence a total annuity to be enjoyed during the whole period of life, is expressed by $\frac{l_{x1}v+l_{x2}v^2 \dots l_zv^{z-x}}{l_x}$. Now, were we to stop here, each annuity could only be obtained after summing up *all* the terms of the series in the numerator, and the total then divided by l_x: an operation by far too laborious; but by multiplying both the numerator and the denominator in the last fraction by v^x (which, as stated before, does not change the value of the fraction), we obtain $a_x=\frac{l_{x1}v^{x1}+l_{x2}v^{x2} \dots + l_zv^z}{l_xv^x}$. From the construction of the D and N columns it will be seen, that $D_x = l_x\, v^x$; thus the last-named fraction becomes $=\frac{D_{x1}+D_{x2} \dots + D_z}{D_x}$. It will further be observed, that N_x expresses the total of the whole D series, from x and upwards to the end of life; so that $a_x=\frac{N_{x+1}}{D_x}$. Now, suppose a deferred annuity be required on a life aged x, not to commence before the expiration of m years, all we have to do is, to reject the first m terms of the numerator in the fraction; and since the entire series of the numerator is N_{x1}, the series less the *first* term $= N_{x2}$, the series less the first *two* terms $= N_{x3}$, and the series less m terms $= N_{x1+m}$; it follows that $\frac{N_{x1+m}}{D_x}$ expresses an annuity on a life aged x deferred m years. And since a temporary, together with a deferred, form a whole-life annuity, it is manifest that a temporary annuity for the next m years $=\frac{N_{x1}-N_{x1+m}}{D_x}$. With the same ease an intercepted annuity can be found. Suppose that after m years the annuity is to cease for n years, then the formula becomes $\frac{N_{x1}-N_{x1+m}+N_{x1+m+n}}{D_x}$.—EDITOR.

§ 48. Thus far I have pointed out the mode of calculating annuities by means of the D and N columns in *numbers*. I now proceed to show how the same may be found more conveniently by the medium of logarithms; and further, how to construct auxiliary logarithmic columns for calculating temporary and deferred annuities without a chance of error. It has been shown in the last paragraph, that the probability of x receiving £1 at the end of one year is $\frac{l_{x1}v}{l_x}$; or, by multiplying both the numerator and denominator by v^x, it becomes $\frac{l_{x1}v^x}{l_xv^x} = \frac{D_{x1}}{D_x}$. In like manner, the probability of x receiving £1 at the end of *two* years is $\frac{l_{x2}v^2}{l_x}$, or if multiplied by v^x as above, it becomes $\frac{l_{x2}v^{x2}}{l_xv^x} = \frac{D_{x2}}{D_x}$; and, at the end of m years $= \frac{D_{x+m}}{D_x}$. It will likewise be manifest, that the probability of x_1 receiving £1 at the end of one year is $\frac{D_{x2}}{D_{x1}}$; and by multiplying the value of the first year's payment on the life x by the value of the first year's payment on the life x_1, we have $\frac{D_{x1}}{D_x} \times \frac{D_{x2}}{D_{x1}} = \frac{D_{x2}}{D_{x1}}$ as before. Let us therefore suppose that a table of logarithmic differences be formed from a logarithmic D column, subtracting always the logarithm of the younger D from the next older one (which in fact implies, that each D in numbers be divided by its next younger D), and let us designate such differences (according to the Carlisle table) d_{103}, d_{102}, d_{101}, &c., so that $\lambda D_{104} - \lambda D_{103} = d_{103}$; $\lambda D_{103} - \lambda D_{102} = d_{102}$; and $\lambda D_{x1} - \lambda D_x = d_x$; we shall then have a ready means for computing the logarithms corresponding to the values of all annual payments singly to be received by a life x. This operation will be found as one most simple. It has been demonstrated before, that the value of the first annual payment on a life x is $\frac{D_{x1}}{D_x}$, the same on a life $_{x1}$ is $\frac{D_{x2}}{D_{x1}}$; and if both these values be multiplied into one another, the product $\frac{D_{x2}}{D_x}$ is the value of the second year's payment on the life x; or, by means of the logarithmic differences we have d_x as the logarithm corresponding to the value of the *first* year's payment; $d_x + d_{x1}$ as the logarithm corresponding to the value of the *second* year's payment, and $d_x + d_{x1} + d_{x2} \ldots + d_{103}$, as the value of the last year's payment. Hence, it is only necessary to start with the logarithm d_x, adding the same to d_{x1}, the sum to d_{x2}, again to d_{x3}, and so on to the end of the table. On having thus produced a column of logarithms, their values in numbers are then to be taken out from an ordinary table of logarithms, and their gradual summations will give the values of temporary annuities for any number of years, to the whole extent of the mortality table. Or, by placing the total annuity at the head of the column (as in col. iv. of Tab. I.), and diminishing the same continually by the values of the annual payments, the remainders form *deferred* annuities, and *vice versa*, in Tab. II., as illustrated by figures on next page.

The necessity of multiplying both the numerator and denominator by v^x will be obvious from the following reasoning. The formula for a_x (as demonstrated in § 31) is $l_{x1}v + l_{x2}v^2 + l_{x3}v^3 \dots + l_n v^{n-x}$, divided by l_x. Thus in each consecutive term in the numerator of the series, the power of v rises gradually with unity. Now, suppose a table containing the numbers living at each age form the only means at our disposal for calculating annuities; our task, according to the above formula, would then be reduced to the following process. Let an annuity be sought on a life aged 50? Agreeably with the above formula, we shall have to form a series, to consist of not less than fifty-four terms in the denominator, commencing with $l_{51}v + l_{52}v^2 + l_{53}v^3 \dots$ and concluding with $l_{104}\,v^{54}$. This whole series will then have to be divided by l_{50}. Suppose such series to be already calculated, its results could be of no benefit in calculating an annuity on a life aged 49; since for the series of the latter, the numerator would commence with $l_{50}v + l_{51}v^2 + l_{52}v^3 \dots$ and end with $l_{104}\,v^{55}$ — independently of the latter requiring one extra term. — In order to render the former series also available for the latter case, the total of all the terms in the former series will require to be multiplied by v — besides one extra term: viz. $l_{50}v$, to be added to the series — before it be divided by l_{49}. In like manner, were the above series to be used for any age $50-m$, the total of its terms would (independently of the extra new terms necessary) have to be multiplied by v^m. It is also clear, that if the same series is to be applied to age 51, that is, one year older, its total, after the rejection of the first term $l_{51}v$, instead of being multiplied, will have to be *divided* by v. Or were the 50-series to be used for any age $50+m$, after rejecting the first m terms, the total of its numerator would require to be divided by v^m; or, more rationally, instead of dividing the numerator, the denominator; viz. l_{50+m}, might be *multiplied* by v^m. Now, since it is desirable that there should be one series only, as a mutual basis, for all cases, and to be worked always with uniformity, the one best adapted for the purpose can be no other than that applicable to age 0; that is, to one just born; and which, according to the Carlisle mortality, must consist of no less than 104 terms: viz., $l_1v + l_2v^2 + l_3v^3 \dots + l_{104}\,v^{104}$. Such series will suffice for all older ages; and after rejecting the first x terms, may be used as a fixed basis for any age $0+x$: always multiplying the denominator l_x by v^x. It is the numerator of this series that constitutes the D column, D representing Denominator. Once having fixed upon the zero-series, all that remained to be done was to form totals of the progressive number of terms from 1 to 104, which, for the sake of brevity, were symbolized by the letter N, meaning Numerator. Hence $\frac{N_{x1}}{D_x}$ means nothing else than the division of the total formed of all the D values from D_{x1} to D_{104}, by D_x. Such fraction, on having its v^x cancelled, both in the Nunerator and Denominator, must always be equal to $l_{x1}v + l_{x1}v^2 \ldots + l_{104}v^{104-x}$. The student will thus perceive, that the construction of the D and N columns are both indispensable in the calculation of life annuities and reversions. — Editor.

Specimen of Table I. for Temporary Annuities.

Age.	D_x	λD_x	d_x	I.	II.	III.	IV.
				Values of Annual Payments:		Temporary Annuities.	Deferred Annuities.
				in Logarithms.	in Numbers.		
0	10000·000	4·0000000	1·9145845				
1	8214·563	3·9145845	·9506649	1·9506649	·8926165	0·8926165	20·0842900
2	7332·454	·8652494	·9580122	·9086771	·8103583	1·7029748	19·1916735
3	6656·740	·8232616	·9703634	·8790405	·7569035	2·4598783	18·3813152
4	6217·632	·7936250	·9745062	·8535467	·7137510	3·1736293	17·6244117
	. . .						
	. . .						
	. . .						
100	·4682956	1·6705200	·8780183	5·6339538	·0000430	20·0842372	0·0000958
101	·3536212	·5485383	·8410348	·4749886	·0000298	20·0842670	·0000528
102	·2452297	·3895731	·7653140	·2403026	·0000174	20·0842844	·0000230
103	·1428523	·1548871	·5100415	6·7503441	·0000056	20·0842900	·0000056
104	·0462305	2·6649286					

Specimen of Table II. for Deferred Annuities.

Age.	D_x	λD_x	$co\text{-}d_x$	Values of Annual Payments:		Deferred Annuities.	Temporary Annuities.
				in Logarithms.	in Numbers.		
104	·0462305	2·6649286	. . .	6·7503441	·0000056	·0000056	20·0842900
103	·1428523	1·1548871	0·4899585	5·2403026	·0000174	·0000230	20·0842844
102	·2452297	·3895731	·2346860	·4749886	·0000298	·0000528	20·0842670
101	·3536212	·5485383	·1589653	·6339539	·0000430	·0000958	20·0842372
100	·4682956	·6705200	·1219817	·7559356	·0000570	·0001528	3·1736293
	. . .	. . .					
	. . .	. . .					
	. . .	. . .					
4	6217·632	3·7936250	·0254939	1·8790405	·7569035	18·3813152	2·4598783
3	6656·740	·8232616	·0296366	·9086771	·8103583	19·1916735	1·7029748
2	7332·454	·8652494	·0419877	·9506649	·8926165	20·0842900	0·8926165
1	8214·563	·9145845	·0493351	·0000000			
0	10000·000	4·0000000	·0854155				

§ 49. It now only remains necessary to substantiate what has been stated at the outset of last section — viz., that the above mode of computing is free from error. It has been shown before, that d_x is the value of the first year's payment on a life x; $d_x + d_{x1}$, the value of the second year's payment; $d_x + d_{x1} + d_{x2}$, the value of the third year's payment; and d_{103} the value of the last year's payment. Consequently, the correctness of the second[1] logarithm may be verified by the difference of log. D_{x2} — log. D_x, which must consist of the same value. Similarly, the last logarithmic result may be verified by the difference of log. D_{103} — log. D_x; so that, in all cases, the correctness of the entire series of the logarithms produced may be tested by the last result of the series, and must invariably agree with log. D_{103} — log. D_x. If, therefore, the last result be found correct, it may be assumed that all the foregoing logarithms are also correct. But suppose it be found incorrect, then, one or other of the intermediate logarithms may be checked by means of log. D_{x+q} — log. D_x; q in this instance indicating the doubted logarithm in question. Such test will in fact rarely be required, inasmuch as the revision of the summations themselves will be found easy enough: since an error will seldom exceed one figure, which can always be detected with facility. Having now shown the mode of verifying the result of the logarithmic column for the values of each year's expectation of receiving £1; the mode of checking the second column, namely, that which contains the values in numbers, follows next. It is manifest that the total of the annual values, to the end of life, is the annuity on the whole life. Now, since the value of $a_x = \frac{N_{x1}}{D_x}$, we can readily test the total sum of all such annual values, previous to forming the third column, which is to contain the temporary annuities. If, therefore, the total agree with $\frac{N_{x1}}{D_x}$, then, the third column may be proceeded with, by adding the first value to the second, the sum to the third, and so on consecutively to the last result, which must necessarily also agree with a_x. In case of disagreement, one or other of the intermediate results can be checked by the formula $\frac{N_{x1} - N_{x+q}}{D_x}$. — EDITOR.

COROLLARY VI.[2]

§ 50. If the annuity is to be enjoyed for a term *certain* ($=n$), and after that during the continuance of any given lives, its value will be expressed by the following series, $v + v^2 + v^3 + \ldots\ldots v^n +$

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[1] It must be borne in mind, that the first logarithm *written* is in fact the second of the series, the first one being already found in the specimen above. — EDITOR.

[2] Morgan, Prob. 15.

$\frac{1}{l_x\, l_y\, l_z}(v^{1+n}\, l_{x1+n}\, l_{y1+n}\, l_{z1+n}+v^{2+n}\, l_{x2+n}\, l_{y2+n}\, l_{z2+n} \ldots)$. But, the first n terms of this series, or $v+v^2+v^3 \ldots v^n$ are equal to $\frac{1-(1+r)^{-n}}{r}$, or to the present value of an annuity of £1 per annum for n years; and the remaining part of the series is, by the third corollary, equal to $a_{x\cdot y\cdot z(m}$: whence the following rule—

§ 51. *To the present value of an annuity certain for the given term, add the value of an annuity on the given lives deferred for that term: the sum will be the answer required.*

For examples of the use and application of this corollary, see Question XXII. in Chapter XII.

COROLLARY VII.

§ 52. If, in the general expression for the value of an annuity on a single life A, we suppose r to vanish, or that money does not bear any interest; then will such expression be reduced to the series $\frac{1}{l_x}(l_{x1}+l_{x2}+l_{x3}$, &c.), and which is the value of an annuity on the life A, considered as a *yearly* annuitant without interest of money: whence the following *rule.* Divide the sum of all the living, at every age after the age of the given life, by the number of persons living at that age; and the quotient will be the value required.

In a subsequent chapter, I shall show that we must add $\frac{1}{4}$ or $\frac{3}{8}$ to the value thus found, in order to obtain the value of a similar annuity on the life A considered as a *half yearly* or *quarterly* annuitant:[1] but when the life is considered as a *momently* annuitant, we must add $\frac{1}{2}$ to such value. In this last case the required value coincides with the sums of the probabilities that such life will attain to the end of the first, second, third, &c., *moments*, from the present time to the end of its possible existence: and is the same with what is denominated, by writers on this subject, the *Expectation of Life;* or, the number of years which, taking lives of the same age one with another, any one of those lives may be considered as sure of enjoying; those who live *beyond* that period, enjoying as much *more* in proportion to their number, as those who *fall short* of it enjoy *less*. Consequently, the rule for finding the Expectation of Life will be as follows—

§ 53. *Divide the sum of all the living, at every age after the age of the given life, by the number of persons living at that age; half unity added to the quotient will be the value required.*

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[1] See Chapter X.

§ 54. As we may sometimes have occasion to determine this value, in order to compare the probabilities of life, or the values of annuities, according to different tables of observations, as well as for various purposes of approximation to be explained hereafter, I have here inserted the rule for finding such values according to any table of observation.[1]

PROBLEM II.[2]

§ 55. To find the value of an annuity granted upon the longest of any number of lives; that is, for as long as any one of them is in existence.

SOLUTION.

Let A, B, C, be the lives upon which the annuity is granted, and let the probability of each life continuing 1, 2, 3, &c., years, be as denoted in § 23: then, the probability that some one or other of these will live to the end of the first year will (by § 26) be expressed by $1 - \left[\left(1-\frac{l_{x1}}{l_x}\right)\left(1-\frac{l_{y1}}{l_y}\right)\left(1-\frac{l_{z1}}{l_z}\right)\right] = \frac{l_{x1}}{l_x} + \frac{l_{y1}}{l_y} + \frac{l_{z1}}{l_z} - \left(\frac{l_{x1}\,l_{y1}}{l_x\,l_y} + \frac{l_{x1}\,l_{z1}}{l_x\,l_z} + \frac{l_{y1}\,l_{z1}}{l_y\,l_z}\right) + \frac{l_{x1}\,l_{y1}\,l_{z1}}{l_x\,l_y\,l_z}$; which, being multiplied by v, or the present value of £1 certain to be received at the end of one year, will give the present value of the first year's rent, or the *expectation* of receiving such sum on the contingency that any one of the lives continues to the end of the first year. In like manner, since the probability that some one or other of the lives will live to the end of the second year is $1 - \left[\left(1-\frac{l_{x2}}{l_x}\right)\left(1-\frac{l_{y2}}{l_y}\right)\left(1-\frac{l_{z2}}{l_z}\right)\right] = \frac{l_{x2}}{l_x} + \frac{l_{y2}}{l_y} + \frac{l_{z2}}{l_z} - \left(\frac{l_{x2}\,l_{y2}}{l_x\,l_y} + \frac{l_{x2}\,l_{z2}}{l_x\,l_z} + \frac{l_{y2}\,l_{z2}}{l_y\,l_z}\right) + \frac{l_{x2}\,l_{y2}\,l_{z2}}{l_x\,l_y\,l_z}$; it follows, that if this expression be multiplied by v, or the present value of £1 certain to be received at the end of two years, it will give the present value of the second year's rent, or the *expectation* of receiving such sum on the contingency that some one or other of the lives will live to the end of the second year. The same method of reasoning will apply to the present value of the third year's rent, or the *expectation* of receiving such sum, on the contingency that some one or other of the lives will live to the end of the third year.

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[1] I would here observe, that according to the hypothesis of M. De Moivre, which will be explained hereafter, the expectation of any single life is equal to *half the complement* of that life: consequently, the Complement of any life is equal to twice the Expectation of that life. This mode of expression is sometimes used, even when speaking of values *deduced from real observations.*

[2] Simpson, Prob. 2. De Moivre, Prob. 4 and 5. Dodson, vol. ii. Ques. 76 to 86.; vol. iii. Ques. 14.

And so on for all the subsequent years, to the utmost extent of human life: the sum of all which expectations,[1] or the series,

$$
\begin{aligned}
& v.\left[\frac{l_{x1}}{l_x}+\frac{l_{y1}}{l_y}+\frac{l_{z1}}{l_z}-\left(\frac{l_{x1}\,l_{y1}}{l_x\,l_y}+\frac{l_{x1}\,l_{z1}}{l_x\,l_z}+\frac{l_{y1}\,l_{z1}}{l_y\,l_z}\right)+\frac{l_{x1}\,l_{y1}\,l_{z1}}{l_x\,l_y\,l_z}\right] \\
+\ & v^2\left[\frac{l_{x2}}{l_x}+\frac{l_{y2}}{l_y}+\frac{l_{z2}}{l_z}-\left(\frac{l_{x2}\,l_{y2}}{l_x\,l_y}+\frac{l_{x2}\,l_{z2}}{l_x\,l_z}+\frac{l_{y2}\,l_{z2}}{l_y\,l_z}\right)+\frac{l_{x2}\,l_{y2}\,l_{z2}}{l_x\,l_y\,l_z}\right] \\
+\ & v^3\left[\frac{l_{x3}}{l_x}+\frac{l_{y3}}{l_y}+\frac{l_{z3}}{l_z}-\left(\frac{l_{x3}\,l_{y3}}{l_x\,l_y}+\frac{l_{x3}\,l_{z3}}{l_x\,l_z}+\frac{l_{y3}\,l_{z3}}{l_y\,l_z}\right)+\frac{l_{x3}\,l_{y3}\,l_{z3}}{l_x\,l_y\,l_z}\right] \\
+\ & \ldots\ldots\ldots
\end{aligned}
$$

will be the total present value of the annuity required.

§ 56. But, the first collateral column in this general expression denotes the value of an annuity on the life A; the second denotes the value of an annuity on the life B; the third denotes the value of an annuity on the life C, &c. In like manner, the fourth, fifth, and sixth collateral columns denote the values of an annuity on the joint lives AB, AC, BC, &c., respectively; and so on. Whence, if we substitute for these respective values the characters mentioned in Prob. I. cor. 1, the whole expression, or the sum of all the terms in the above series, will become equal to $a_x+a_y+a_z-(a_{x\cdot y}+a_{x\cdot z}+a_{y\cdot z})+a_{x\cdot y\cdot z}$: consequently, the value of an annuity, to continue as long as any one of the given lives is in existence, is equal to the sum of the values of an annuity on all the single lives, *minus* the sum of the values of an annuity on all the joint lives combined, two and two, *plus* the sum of the values of an annuity on all the joint lives combined, three and three, *minus* the sum of the values of an annuity on all the joint lives combined, four and four and so on. Therefore, when the values of an annuity on the single and joint lives are given, the value of an annuity on the longest life may be very easily determined: and for the sake of a more convenient reference, I shall take L to denote the value found by this rule; the *number* of lives, which it is intended to represent, being always explained when the character is used.

For examples of the use and application of this problem, see Questions VIII. and IX. in Chapter XII.

COROLLARY I.[2]

§ 57. If the lives are all equal, or of the same age A, and their number be represented by n; then the probability that some one or other of such lives will continue to the end of the first, second, third, &c., years respectively, will be severally denoted by

---

[1] When the ages of the given lives differ, the number of terms composing those several series will also differ; but I would here observe, once for all, that in every case of combined lives, the terms must be continued to the extinction of the oldest life involved in the combination; or universally, *each* series must be continued till the terms vanish, which will happen when either of the lives involved in such series has arrived at the extremity of human life.

[2] Dodson, vol. ii. Ques. 78, 85 and 86. Vol. iii. Ques. 11.

$$1-\left(1-\frac{l_{x1}}{l_x}\right)^n, 1-\left(1-\frac{l_{x2}}{l_x}\right)^n, 1-\left(1-\frac{l_{x3}}{l_x}\right)^n, \text{ \&c.}$$

Let these quantities be severally expanded by means of the binomial theorem, in order to reduce them to simple terms, and then be respectively multiplied into the present value of £1 due at the end of those years; the sum of them, or the series

$$\begin{aligned}
&v\left[n.\frac{l_{x1}}{l_x}-\frac{n.(n-1)}{2}.\frac{l_{x1}\,l_{x1}}{l_x\,l_x}+\frac{n.(n-1)}{2}.\frac{(n-2)}{3}.\frac{l_{x1}\,l_{x1}\,l_{x1}}{l_x\,l_x\,l_x}-\ldots\right]\\
+\,&v^2\left[n.\frac{l_{x2}}{l_x}-\frac{n.(n-1)}{2}.\frac{l_{x2}\,l_{x2}}{l_x\,l_x}+\frac{n.(n-1)}{2}.\frac{(n-2)}{3}.\frac{l_{x2}\,l_{x2}\,l_{x2}}{l_x\,l_x\,l_x}-\ldots\right]\\
+\,&v^3\left[n.\frac{l_{x3}}{l_x}-\frac{n.(n-1)}{2}.\frac{l_{x3}\,l_{x3}}{l_x\,l_x}+\frac{n.(n-1)}{2}.\frac{(n-2)}{3}.\frac{l_{x3}\,l_{x3}\,l_{x3}}{l_x\,l_x\,l_x}-\ldots\right]\\
+\,&v^n\quad.\quad.\quad.
\end{aligned}$$

will be the total present value of the annuity required.

§ 58. Whence (if we take $a_{x\cdot x}$, $a_{x\cdot x\cdot x}$, to denote the value of an annuity on two, three, equal joint lives) it is manifest that the value of an annuity on the longest of any number of lives, all of age x, will be

$$na_x-\frac{n.(n-1)}{2}\,a_{x\cdot x}+\frac{n.(n-1)}{2}.\frac{(n-2)}{3}\,a_{x\cdot x\cdot x}$$

Therefore, if the number of lives be two, this expression will become $2a_x-a_{x\cdot x}$; if three, it will become $3a_x-3a_{x\cdot x}+a_{x\cdot x\cdot x}$; if four, it will become $4a_x-6a_{x\cdot x}+4a_{x\cdot x\cdot x}-a_{x\cdot x\cdot x\cdot x}$; and so on.

For examples of the use and application of this corollary, see Question VIII. in Chapter XII.

COROLLARY II.[1]

§ 59. If we wish to determine the value of an annuity on the longest of any number of lives, *Deferred* for a given term ($=m$), it will be evident (from what has been said in Prob. I. cor. 3) that the several perpendicular series in last page must not commence till after the mth term; and thence be continued to the utmost extent of human life. Consequently, the required value of such deferred annuity will be equal to $a_{x(m}+a_{y(m}+a_{z(m}-a_{x\cdot y(m}-a_{x\cdot z(m}-a_{y\cdot z(m}+a_{x\cdot y\cdot z(m}$: an expression which I shall, for the sake of a more convenient reference, denote by $L_{(m}$. Now, if for each of these several quantities we substitute its corresponding value, agreeably to the principles laid down in page 35, the formula denoted by $L_{(m}$ will become equal to

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[1] Dodson, vol. iii. Ques. 13, 15.

$$v^m\left(a_{x+m}\frac{l_{x+m}}{l_x} + a_{y+m}\frac{l_{y+m}}{l_y} + a_{z+m}\frac{l_{z+m}}{l_z} - \left[a_{x\cdot y|m}\frac{l_x\,l_{y|m}}{l_x\,l_y} + a_{x\cdot z|m}\frac{l_x\,l_{z|m}}{l_x\,l_z} + a_{y\cdot z|m}\frac{l_y\,l_{z|m}}{l_y\,l_z}\right] + a_{x\cdot y\cdot z|m}\frac{l_x\,l_y\,l_{z|m}}{l_x\,l_y\,l_z}\right):$$

which is an expression more convenient for practice; and from either of which we deduce the following rule.

§ 60. *Substitute the values of deferred annuities on the single and joint lives, in the general rule in the problem, instead of the values of annuities on the whole continuance of those lives; and proceed with these substituted values according to the directions given in such rule: the result will be the answer required.*

For examples of the use and application of this corollary, see Question XI. in Chapter XII.

COROLLARY III.

§ 61. If an annuity on the longest of any number of lives, deferred for any given term, depends on the *joint* continuance of all those lives to the end of that term; the value of it will be equal to $L \times v^m \frac{l_x\,l_y\,l_{z|m}}{l_x\,l_y\,l_z}$: that is, equal to the value of an annuity on the longest of the same number of lives, each older by the given term than the given lives, multiplied by the expectation that the *joint* lives shall receive £1 at the end of that term: and this case must be carefully distinguished from the one mentioned in the preceding corollary.

For examples of the use and application of this corollary, see the Scholium to Question XI. in Chapter XII.

COROLLARY IV.

§ 62. Having found, by means of the second corollary here given, the value of a deferred annuity on the longest of any number of lives, we may easily determine the value of a *Temporary*[1] annuity on the longest of such lives: it being nothing more than the difference between the value of such deferred annuity and the value of a similar annuity on the whole continuance of the lives, as found by the problem. That is, the formula $L - L_{(m}$ will in all cases denote the value of such temporary annuity: similar to what takes place with respect to joint lives, as already explained in page 36.

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[1] That is, of an annuity which is to commence immediately, but to continue only during a given number of years ($= m$) which is less than that to which it is possible *either* of the given lives may extend. Therefore, such term must not be greater than the difference between the age of the *youngest* life involved, and the age of the oldest life in the table of observations: for, in such case, the value is found by the Problem. See what has been said respecting single and joint lives in the note in page 36.

For examples of the use and application of this corollary, see Question XII. in Chapter XII.

PROBLEM III.[1]

§ 63. To find the value of an annuity granted upon any number of lives, but to continue only as long as any number ($= n$) of them are in being together.

SOLUTION.

Let the lives on which the annuity is granted be A, B, C: and the probability of each life continuing 1, 2, 3, &c. years, be as denoted in § 23. Now, if we confine this case to that of an annuity granted upon three lives, and to continue as long as any *two* of them (viz. AB, AC, or BC) are in being together,[2] it is evident that the chance of an annuity being received in any one year will depend upon either of these four different events: 1st, That all the lives continue in being together to the end of that year, the probability of which in the first year is $\frac{l_x l_y l_{z|1}}{l_x l_y l_z}$: 2d, That A and B are then alive and C dead, the probability of which in the same year is $\frac{l_x l_{y|1}}{l_x l_y}(1-\frac{l_{z1}}{l_z})$: 3rd, That A and C are then alive, and B dead, the probability of which in the same year is $\frac{l_x l_{z|1}}{l_x l_z}(1-\frac{l_{y1}}{l_y})$: 4th, That B and C are then alive and A dead, the probability of which in the same year is $\frac{l_y l_{z|1}}{l_y l_z}(1-\frac{l_{x1}}{l_x})$. The sum of all these chances, therefore, or $\frac{l_x l_y l_{z|1}}{l_x l_y l_z} + \frac{l_x l_{y|1}}{l_x l_y}(1-\frac{l_{z1}}{l_z}) + \frac{l_x l_{z|1}}{l_x l_z}(1-\frac{l_{y1}}{l_y}) + \frac{l_y l_{z|1}}{l_y l_z}(1-\frac{l_{x1}}{l_x}) = \frac{l_x l_{y|1}}{l_x l_y} + \frac{l_x l_{z|1}}{l_x l_z} + \frac{l_y l_{z|1}}{l_y l_z} - 2(\frac{l_x l_y l_{z|1}}{l_x l_y l_z})$, being multiplied by v, will give the present value of the first year's rent, or the *expectation* of receiving such sum on the contingency that any two of the original lives will outlive the first year. By a similar method of reasoning it will be found that $\frac{l_x l_{y|2}}{l_x l_y} + \frac{l_x l_{z|2}}{l_x l_z} + \frac{l_y l_{z|2}}{l_y l_z} - 2(\frac{l_x l_y l_{z|2}}{l_x l_y l_z})$, multiplied by v, will give the present value of the second year's rent, or the *expectation* of receiving such sum on the contingency that any two of the original lives will outlive the second year. Also that $\frac{l_x l_{y|3}}{l_x l_y} + \frac{l_x l_{z|3}}{l_x l_z} + \frac{l_y l_{z|3}}{l_y l_z} - 2(\frac{l_x l_y l_{z|3}}{l_x l_y l_z})$, multiplied by v, will give the present value of the third year's rent; and so on, for all the subsequent years to the utmost extremity of human life; the sum of all which expectations, or the series

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[1] Simpson, Prob. 3. Morgan, Prob. 7. Dodson, vol. ii. Ques. 87.

[2] All the cases of *two* lives, and likewise the *remaining cases of three lives*, may be solved by means of the preceding problems.

$$\begin{array}{l} v \left[\frac{l_x\, l_{y|1}}{l_x\, l_y} + \frac{l_x\, l_{z|1}}{l_x\, l_z} + \frac{l_y\, l_{z|1}}{l_y\, l_z} - 2\left(\frac{l_x\, l_y\, l_{z|1}}{l_x\, l_y\, l_z}\right)\right] \\ v^2 \left[\frac{l_x\, l_{y|2}}{l_x\, l_y} + \frac{l_x\, l_{z|2}}{l_x\, l_z} + \frac{l_y\, l_{z|2}}{l_y\, l_z} - 2\left(\frac{l_x\, l_y\, l_{z|2}}{l_x\, l_y\, l_z}\right)\right] \\ v^3 \left[\frac{l_x\, l_{y|3}}{l_x\, l_y} + \frac{l_x\, l_{z|3}}{l_x\, l_z} + \frac{l_y\, l_{z|3}}{l_y\, l_z} - 2\left(\frac{l_x\, l_y\, l_{z|3}}{l_x\, l_y\, l_z}\right)\right] \\ + \quad . \quad . \quad . \quad . \quad . \quad . \end{array}$$

will be the total present value of the annuity required.

§ 64. But, if we take the value of each perpendicular series, as in the last problem, and substitute the characters given in Prob. I. cor. 1, the above expression will become $a_{x \cdot y} + a_{x \cdot z} + a_{y \cdot z} - 2\, a_{x \cdot y \cdot z}$: whence the following rule for this particular case. From the sum of the values of an annuity on each pair of joint lives, take twice the value of an annuity on the three joint lives; the difference will be the required value of an annuity to continue as long as any two of the lives are in being together.

For examples of the use and application of this problem, see Question X. in Chapter XII.

§ 65. By a similar method of proceeding, we might ascertain the value of an annuity granted on four lives, but to continue only as long as any *two* or *three* of them are in being together (the other cases of such lives being already solved by the two preceding problems); and, in general, the value of an annuity on any number of lives, to continue only as long as any number of them are in being together. As it rarely happens, however, that more than three lives are concerned in any practical cases, I shall not trouble the reader with the steps of the process, but shall merely state the result of the investigation in one general formula, which will comprehend in one view all the possible cases mentioned in the three problems here given.[1] Let the sum of the values of an annuity on every n joint lives be denoted by S; on every $(n + 1)$ joint lives, by S_1; on every $(n + 2)$ joint lives, by S_2; on every $(n + 3)$ joint lives, by S_3, &c.:[2] then will the value of an annuity granted upon any number of lives, to continue only as long as any n of them are in being together, be expressed by

$$S - n.\ S_1 + \frac{n}{1} \cdot \frac{n+1}{2} .\ S_2 - \frac{n}{1} \cdot \frac{n+1}{2} \cdot \frac{n+1}{3} .\ S_3 + \&c.$$ [3]

---

[1] It will be readily seen, that Problems I. and II. are only particular cases of this third problem.

[2] That is, on every combination which can be made of the given lives, by combining n, $(n+1)$, $(n+2)$, $(n+3)$, &c. lives together at a time.

[3] This formula is taken from a pamphlet, written by Dr. Waring, *On the Principles of Translating Algebraic Quantities into Probable Relations.* (1792).—EDITOR.

COROLLARY I.

§ 66. If the annuity is not to continue during the whole period of the given lives, but is either *Deferred* or *Temporary*, we must pursue the same method of reasoning which has been adopted in Prob. II., cor. 2 and 4: whereby it will appear that the value of a deferred annuity, to continue as long as any two out of three given lives are in being, will be expressed by $a_{x\cdot y(m} + a_{x\cdot z(m} + a_{y\cdot z(m} - 2\,(a_{x\cdot y\cdot z(m}$. Therefore, if we substitute the values of deferred annuities on the given lives (in the formula in § 64), instead of the values of annuities on the whole continuance of those lives, we shall obtain the value of the deferred annuity depending on the contingency alluded to in the problem.

And this value, being subtracted from the value of a similar annuity on the whole continuance of the lives, will give the value of a similar *Temporary* annuity on any two out of three lives.

COROLLARY II.

§ 67. If such deferred annuity, however, depends on the *joint* continuance of all the given lives to the end of the given term, the value of it will be equal to the value of a similar annuity on the same number of lives each older by the given term than such lives, multiplied by the expectation that the *joint* lives shall receive £1 at the end of that term: and this case must be carefully distinguished from that mentioned in the first part of the preceding corollary.

CHAPTER III.

ON REVERSIONS.

§ 68. A REVERSIONARY life annuity is a term applied to such periodical sums of money, depending on any given lives, as are not payable till after a given term, or till after the extinction of any other given lives. Of the former kind are all *deferred* life annuities, mentioned in Prob. I. cor. 3: but it is my intention now to treat only of the latter kind; and I would here observe, that I shall continue to designate the former by the title of Deferred life annuities, applying the term *Reversionary* life annuities to such life annuities only as are not to be enjoyed till after the extinction of some other life. The several cases relating to this subject may be comprised in the four following problems.

PROBLEM IV.[1]

§ 69. To find the value of an annuity depending on any number of joint lives A B C, after the extinction of any number of other joint lives P Q R.[2]

SOLUTION.

Let the probability of the joint lives A B C continuing 1, 2, 3, &c. years, be respectively denoted by $\frac{l_x\, l_y\, l_{z|1}}{l_x\, l_y\, l_z}$, $\frac{l_x\, l_y\, l_{z|2}}{l_x\, l_y\, l_z}$, $\frac{l_x\, l_y\, l_{z|3}}{l_x\, l_y\, l_z}$, &c., as in § 24; and let the probability of the joint lives P Q R continuing 1, 2, 3, &c. years, be denoted by $\frac{l_p\, l_q\, l_{r|1}}{l_p\, l_q\, l_r}$, $\frac{l_p\, l_q\, l_{r|2}}{l_p\, l_q\, l_r}$, $\frac{l_p\, l_q\, l_{r|3}}{l_p\, l_q\, l_r}$, &c., respectively. Now, the chance which the joint lives A B C have of receiving the annuity in one year, will depend upon their living to the end of that year, and on the joint lives P Q R becoming extinct before the end of that period. The probability of this event happening in the first year is $\frac{l_x\, l_y\, l_{z|1}}{l_x\, l_y\, l_z}(1 - \frac{l_p\, l_q\, l_{r|1}}{l_p\, l_q\, l_r})$; which being multiplied by v, will give the present value of the first year's rent, or the expectation of receiving such sum at the end of the first year. By a similar method of reasoning, it will be found that $\frac{l_x\, l_y\, l_{z|2}}{l_x\, l_y\, l_z}(1 - \frac{l_p\, l_q\, l_{r|2}}{l_p\, l_q\, l_r})$, multiplied by v^2, will denote the present value of the second year's rent, and that $\frac{l_x\, l_y\, l_{z|3}}{l_x\, l_y\, l_z}(1 - \frac{l_p\, l_q\, l_{r|3}}{l_p\, l_q\, l_r})$, multiplied by v^3, will denote the present value of the third year's rent; and so on for all the subsequent years to the utmost extent of human life: the sum of all which terms, or the series

$$\begin{aligned} & v\left(\frac{l_x\, l_y\, l_{z|1}}{l_x\, l_y\, l_z} - \frac{l_x\, l_y\, l_z\, l_p\, l_q\, l_{r|1}}{l_x\, l_y\, l_z\, l_p\, l_q\, l_r}\right) \\ + \; & v^2\left(\frac{l_x\, l_y\, l_{z|2}}{l_x\, l_y\, l_z} - \frac{l_x\, l_y\, l_z\, l_p\, l_q\, l_{r|2}}{l_x\, l_y\, l_z\, l_p\, l_q\, l_r}\right) \\ + \; & v^3\left(\frac{l_x\, l_y\, l_{z|3}}{l_x\, l_y\, l_z} + \frac{l_x\, l_y\, l_z\, l_p\, l_q\, l_{r|3}}{l_x\, l_y\, l_z\, l_p\, l_q\, l_r}\right) \\ + \; & \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \end{aligned}$$

will be the total present value of the annuity required. But the sum of these two perpendicular series is evidently equal to $a_{x \cdot y \cdot z} - a_{x \cdot y \cdot z \cdot p \cdot q \cdot r}$.

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[1] Simpson, Prob. 5.

[2] The lives P Q R are said to be in *possession*, in opposition to A B C, which are said to be in *reversion*.

§ 70. Whence it follows, that if we subtract the value of an annuity on all the joint lives, from the value of an annuity on the joint lives in reversion, the remainder will be the value of the reversionary annuity required.

PROBLEM V.[1]

§ 71. To find the value of an annuity depending on any number of joint lives A B C, after the extinction of the longest of any number of other lives, P Q R.

SOLUTION.

Let the probabilities of the given lives continuing 1, 2, 3, &c., years, be denoted by the same characters as in the last problem. Now the chance which the joint lives A B C have of receiving the annuity at the end of any one year, will depend upon their living to the end of that year, and on all the lives P Q R becoming extinct before the end of that period. The probability of this event happening in the first year is $\frac{l_x\,l_y\,l_{z|1}}{l_x\,l_y\,l_z}(1-\frac{l_{p1}}{l_p})\,(1-\frac{l_{q1}}{l_q})\,(1-\frac{l_{r1}}{l_r})$,[2] which, being multiplied by v, will give the present value of the first year's rent. In like manner, $\frac{l_x\,l_y\,l_{z|2}}{l_x\,l_y\,l_z}(-1\frac{l_{p2}}{l_p})\,(1-\frac{l_{q2}}{l_q})\,(1-\frac{l_{r2}}{l_r})$, multiplied by v^2, and $\frac{l_x\,l_y\,l_{z|3}}{l_x\,l_y\,l_z}(1-\frac{l_{p3}}{l_p})$ $(1-\frac{l_{q3}}{l_q})\,(1-\frac{l_{r3}}{l_r})$, multiplied by v^3, will give the present value of the second and third year's rents respectively: and so on for all the subsequent years, to the utmost extent of human life; the sum of all which terms will be the present value of the annuity required. But these expressions, reduced to their simplest terms, are equal to the series

$$v\,\frac{l_x\,l_y\,l_{z|1}}{l_x\,l_y\,l_z}\left[1-\left(\frac{l_{p1}}{l_p}+\frac{l_{q1}}{l_q}+\frac{l_{r1}}{l_r}\right)+\left(\frac{l_p\,l_{q|1}}{l_p\,l_q}+\frac{l_p\,l_{r|1}}{l_p\,l_r}+\frac{l_q\,l_{r|1}}{l_q\,l_r}\right)-\frac{l_p\,l_q\,l_{r|1}}{l_p\,l_q\,l_r}\right]$$
$$+v^2\,\frac{l_x\,l_y\,l_{z|2}}{l_x\,l_y\,l_z}\left[1-\left(\frac{l_{p2}}{l_p}+\frac{l_{q2}}{l_q}+\frac{l_{r2}}{l_r}\right)+\left(\frac{l_p\,l_{q|2}}{l_p\,l_q}+\frac{l_p\,l_{r|2}}{l_p\,l_r}+\frac{l_q\,l_{r|2}}{l_q\,l_r}\right)-\frac{l_p\,l_q\,l_{r|2}}{l_p\,l_q\,l_r}\right]$$
$$+v^3\,\frac{l_x\,l_y\,l_{z|3}}{l_x\,l_y\,l_z}\left[1-\left(\frac{l_{p3}}{l_p}+\frac{l_{q3}}{l_q}+\frac{l_{r3}}{l_r}\right)+\left(\frac{l_p\,l_{q|3}}{l_p\,l_q}+\frac{l_p\,l_{r|3}}{l_p\,l_r}+\frac{l_q\,l_{r|3}}{l_q\,l_r}\right)-\frac{l_p\,l_q\,l_{r|3}}{l_p\,l_q\,l_r}\right]$$
$$+\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot$$

the sum of which is evidently equal to $a_{xyz}-a_{xyzp}-a_{xyzq}-a_{xyzr}+a_{xyzpq}$ $+a_{xyzpr}+a_{xyzqr}-a_{xyzpqr}$.

§ 72. Whence it follows that the value of a reversionary annuity on any number of joint lives A B C, &c., after the extinction of the longest of any number of other lives P Q R, &c., is equal to the value of an annuity on all the joint lives in reversion; *minus* the sum of the values of an annuity on all the joint lives, arising from the combination of all the joint lives in reversion with each ONE of the other lives; *plus* the sum of the values of an annuity on all the joint lives, arising from the combination of all the joint lives in reversion with each TWO of the other lives; *minus* the sum of the values of an annuity on all the joint lives, arising from the combination of all the joint lives in reversion with each THREE of the other lives; and so on.

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[1] Simpson, Prob. 6.

[2] See Chapter I. § 9.

PROBLEM VI.[1]

§ 73. To find the value of an annuity on the longest of any number of lives A B C, after the extinction of any number of joint lives P Q R.

SOLUTION.

Let the probabilities of the given lives continuing 1, 2, 3, &c., years be denoted by the same characters as in the preceding problems. Now, the chance which any one of the lives A B C has of receiving the annuity at the end of any one year, will depend upon either of them living to the end of that year, and on the extinction of the joint lives P Q R before the end of that period. The probability of this happening in the first year is

$$\left[1-(1-\frac{l_{x1}}{l_x})(1-\frac{l_{y1}}{l_y})(1-\frac{l_{z1}}{l_z})\right](1-\frac{l_p\,l_q\,l_{r|1}}{l_p\,l_q\,l_r}),$$

which, being multiplied by v, will give the present value of the first year's rent. In like manner, it will be found that

$$\left[1-(1-\frac{l_{x2}}{l_x})(1-\frac{l_{y2}}{l_y})(1-\frac{l_{z2}}{l_z})\right](1-\frac{l_p\,l_q\,l_{r|2}}{l_p\,l_q\,l_r}),$$

multiplied by v^2, and that

$$\left[1-(1-\frac{l_{x3}}{l_x})(1-\frac{l_{y3}}{l_y})(1-\frac{l_{z3}}{l_z})\right](1-\frac{l_p\,l_q\,l_{r|3}}{l_p\,l_q\,l_r}),$$

multiplied by v^3, will give the present value of the second and third year's rents respectively; and so on for all the subsequent years, to the utmost extent of human life: the sum of all which terms will be the total present value of the annuity required. But these annual expectations being reduced to their simplest terms, and arranged under each other as in the preceding problems, will form fourteen collateral series, the sum of all which will be found equal to $a_x+a_y+a_z-a_{xy}-a_{xz}-a_{yz}+a_{xyz}-a_{pqrx}-a_{pqry}-a_{pqrz}+a_{pqrxy}+a_{pqrxz}-a_{pqryz}-a_{pqrxyz}$.

§ 74. Whence it follows, that the value of a reversionary annuity, on the longest of any number of lives after any number of joint lives, is equal to the value of an annuity on the longest of all the lives in reversion; *minus* the sum of the values of an annuity on all the joint lives arising from the combination of all the joint lives in possession with each ONE of the other lives; *plus* the sum of the values of an annuity on all the joint lives arising from the combination of all the joint lives in possession with each TWO of the other lives; *minus* the sum of the values of an annuity on all the joint lives arising from the combination of all the joint lives in possession with each THREE of the other lives; and so on.

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[1] Simpson, Prob. 7.

PROBLEM VII.[1]

§ 75. To find the value of an annuity on the longest of any number of lives, A B C, &c., after the extinction of the longest of any number of other lives, P Q R, &c.

SOLUTION.

From what has been already said in the preceding problems, it will be evident that the annuity here alluded to is to continue during the longest of all the given lives A B C, P Q R; and such would be the value of it, were the lives A B C to come into possession immediately. But as they are to receive nothing during the existence of any one of the lives P Q R, the value of an annuity on the longest of their lives must consequently be subtracted.

Whence it follows, that if we subtract the value of an annuity on the longest of the lives in possession from the value of an annuity on the longest of all the lives concerned, the remainder will be the value of the reversionary annuity required.

SCHOLIUM.

§ 76. By the help of these four problems may all the various cases in reversionary annuities be solved. They have been stated at length, in order to give a general view of the subject; but it will readily appear, that the combinations of lives thence arising are much more numerous than occur in any practical cases; and a ready solution may therefore not immediately present itself amidst the multiplicity of symbols. It rarely happens that more than three lives are involved in any questions of this kind: and in order to prevent any difficulty or confusion in referring to the problems for a solution of such questions, I have selected all the possible cases in which not more than three lives are concerned; to which I have added the algebraic solution of the same: where a_x, a_y, a_p, a_q, denote the same as in the preceding problems. The cases are five in number:[1] viz., to find the value of an annuity —

---

[1] Simpson, Prob. 4.

1. On a single life A after another life P: in which case the value of the reversionary annuity is equal to $a_x - a_{xp}$.

2. On a single life A after the longest of two lives P, Q, in which case the value of the reversionary annuity is equal to $a_x - a_{xp} - a_{xq} + a_{xpq}$.

3. On the longest of two lives A, B, after a single life P: in which case the value of the reversionary annuity is equal to $a_x + a_y - a_{xy} - a_{xp} - a_{yp} + a_{xyp}$.

4. On a single life A after two joint lives P Q: in which case the value of the reversionary annuity is equal to $a_x - a_{xpq}$.

5. On two joint lives A B after a single life P: in which case the value of the reversionary annuity is equal to $a_{xy} - a_{xyp}$.

For examples of these several cases, see Questions XIII to XVII in Chapter XII.

COROLLARY I.

§ 77. If the contingency of receiving the annuity is not to continue during the whole period of the given lives, but is either *Deferred*[2] or *Temporary*, we must revert to what has been said in Prob. I. cor. 3 and 4, and substitute the values of such deferred and temporary annuities, deduced from the rules there given, instead of the values of annuities on the whole continuance of those lives; whence by proceeding with these substituted values according to the rules above stated, we shall obtain the required value of the reversionary annuity accordingly.

But this rule may be rendered more convenient for practice by adopting the principles laid down in Prob. II. cor. 2 and 4. Thus, in the first case given in the Scholium, if the contingency is *deferred* for n years, the value of the reversionary annuity will be

$$a_{x(n} - a_{x\cdot p(n} = a_{x+n} \times \frac{v^n l_{x+n}}{l_x} - a_{x\cdot p|n} \times v^n \frac{l_x\, l_{m|n}}{l_x\, l_m}.$$

And this value subtracted from $a_x - a_{xp}$ will give the value of a similar *temporary* reversionary annuity. The same method of solution will apply to the remaining cases given in the Scholium.

For examples of the use and application of this corollary, see Questions XVIII. and XIX. in Chapter XII.

COROLLARY II.[3]

§ 78. If a reversionary annuity — of which the contingency of enjoying is deferred for any number of years—depends on the *joint* continuance of all the lives to the end of that term, the value of it will be equal to the value of a reversionary annuity depending on the same number of lives each older by the given term than the given lives, multiplied by the probability that *all* the joint lives shall continue so long, and also by the present value of £1 due at the end of that term. And this case should be carefully distinguished from the deferred reversionary annuity mentioned in the preceding corollary.

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[1] Simpson, Prob. 13 to 17; and his *Sup.* Prob. 15 to 19; Dodson, vol. ii., Ques. 94, 95, 97, 99, and 101; and vol. iii., Ques. 20, 27, 28 and 29. Also De Moivre, Prob. 7 and 8, for the first and second cases.

[2] Price, Note (D).

[3] Price, Note (C).

For examples of the use and application of this corollary, see the Scholium to Question XIX. in Chapter XII.

CHAPTER IV.

ON SURVIVORSHIPS.

§ 79. In the preceding chapters I have considered the value of annuities as depending on the continuance of any number of lives out of any number of given lives; and also the value of reversionary annuities on any number of lives, after the extinction of any number of other lives. I come now to questions of a mixed nature, where the value of the annuity not only depends on the continuance of the given lives, but also on any survivorship between them. In cases of this kind the annuity is frequently enjoyed in different proportions by the persons on whose lives the same is granted; and therefore they are capable of great variety. The nature and extent of such questions will best appear from the following problems.

PROBLEM VIII.[1]

§ 80. A, B and C agree amongst themselves to purchase an annuity on the longest of their lives, which is to be equally divided between them whilst they are all living; but on the decease of either of them it is to be equally divided between the two survivors, during their joint lives; and then to belong entirely to the last survivor for his life: To find the value of their respective shares, or the proportion which each person ought to contribute towards the purchase.

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[1] Simpson, Prob. 20; also his *Sup.* Prob. 23. De Moivre, Prob. 9. Dodson, vol. iii. Ques. 73.

SOLUTION.

Let the probabilities of the given lives continuing 1, 2, 3, &c. years, be severally denoted by the same characters as in § 23; and let us first determine the share of A. Now the expectation of A, on what he may happen to receive at the end of any one year, may be considered in four parts, as depending on so many different events: 1. A, B and C may be all living, the probability of which at the end of the first year is $\frac{l_x l_y l_{z|1}}{l_x l_y l_z}$, in which case he will receive $\frac{1}{3}$ of the annuity, or $\frac{1}{3}v.$; therefore $\frac{v. l_x l_y l_{z|1}}{3 l_x l_y l_z}$, will be the value of this expectation: 2. A and B may be living and C dead, the probability of which at the end of the first year is $\frac{l_x l_{y|1}}{l_x l_y} (1-\frac{l_{z1}}{l_z})$, in which case he will receive $\frac{1}{2}$ the annuity, or $\frac{1}{2}v.$; therefore $\frac{v\, l_x l_{y|1}}{2 l_x l_y} (1-\frac{l_{z1}}{l_z})$, will be the value of the expectation: 3. A and C may be living and B dead, the probability of which at the end of the first year is $\frac{l_x l_{z|1}}{l_x l_y} (1-\frac{l_{y1}}{l_y})$, in which case he will likewise receive $\frac{1}{2}$ the annuity, or $\frac{1}{2}v$; therefore $\frac{v.\, l_x l_{z|1}}{2.\, l_x l_z} (1-\frac{l_{y1}}{l_y})$, will be the value of this expectation: 4. A may be the only person living, the probability of which at the end of the first year is $\frac{l_{x1}}{l_x} (1-\frac{l_{y1}}{l_y}) (1-\frac{l_{z1}}{l_z})$, in which case he comes in for the whole annuity; therefore $\frac{v.\, l_{x1}}{l_x} (1-\frac{l_{y1}}{l_y}) (1-\frac{l_{z1}}{l_z})$, will be the value of the expectation. And the sum of all these values, or

$$v.\left(\frac{l_{x1}}{l_x} - \frac{l_x l_{y|1}}{2 l_x l_y} - \frac{l_x l_{z|1}}{2 l_x l_z} + \frac{l_x l_y l_{z|1}}{3 l_x l_y l_z}\right)$$

will be the total value of the expectation of A, on what he may happen to receive at the end of the first year.

By pursuing the same steps, it will be found that

$$v^2\left(\frac{l_{x2}}{l_x} - \frac{l_x l_{y|2}}{l_x l_y} - \frac{l_x l_{z|2}}{l_x l_z} + \frac{l_x l_y l_{z|2}}{l_x l_y l_z}\right)$$

will denote the value of his expectation on what he may happen to receive at the end of the second year; and that

$$v^3\left(\frac{l_{x3}}{l_x} - \frac{l_x l_{y|3}}{l_x l_y} - \frac{l_x l_{z|3}}{l_x l_z} + \frac{l_x l_y l_{z|3}}{l_x l_y l_z}\right)$$

will denote the value of his expectation on what he may happen to receive at the end of the third year; and so on to the utmost extent of human life. The sum of all which terms, or the series

$$
\begin{aligned}
& v . \left[\frac{l_{x1}}{l_x} - \frac{l_x \, l_{y\|1}}{2 \, l_x \, l_y} - \frac{l_x \, l_{z\|1}}{2 \, l_x \, l_z} + \frac{l_x \, l_y \, l_{z\|1}}{3 \, l_x \, l_y \, l_z} \right], \\
+ \; & v^2 \left[\frac{l_{x2}}{l_x} - \frac{l_x \, l_{y|2}}{2 \, l_x \, l_y} - \frac{l_x \, l_{z|2}}{2 \, l_x \, l_z} + \frac{l_x \, l_y \, l_{z|2}}{3 \, l_x \, l_y \, l_z} \right], \\
+ \; & v^3 \left[\frac{l_{x3}}{l_x} - \frac{l_x \, l_{y|3}}{2 \, l_x \, l_y} - \frac{l_x \, l_{z|3}}{2 \, l_x \, l_z} + \frac{l_x \, l_y \, l_{z|3}}{3 \, l_x \, l_y \, l_z} \right], \\
+ \; & \quad . \quad . \quad . \quad . \quad . \quad .
\end{aligned}
$$

will be the total value of his interest in the annuity, or the share which he ought to contribute towards the purchase. But the sum of these several perpendicular series is equal to $a_x - \frac{1}{2}a_{xy} - \frac{1}{2}a_{xz} + \frac{1}{3}a_{xyz}$.

§ 81. As to the share of B or C, it is evident that their expectation of receiving the annuity in any one year will depend on the same events, *mutatis mutandis*, as that of A: whence it follows, that by substituting the values thence arising in the general expression above given, we shall have $a_y - \frac{1}{2}a_{xy} - \frac{1}{2}a_{yz} + \frac{1}{3}a_{xyz}$ for the value of B's share; and $a_z - \frac{1}{2}a_{xz} - \frac{1}{2}a_{yz} + \frac{1}{3}a_{xyz}$ for the value of C's share in the given annuity: whence the following rule for either case.

§ 82. *From the value of an annuity on the given life subtract half the sum of the values of an annuity on both the joint lives arising from the combination of the given life with each one of the other; and to the remainder add one-third of the value of an annuity on the three joint lives: the sum will be the answer required.*

§ 83. *Example.* Suppose the three lives to be aged 20, 30, and 40; the rate of interest to be 3 per cent., and the probabilities of living as at *Carlisle*. Then, by referring to the Tables at the end of this work, it will be found that the value of the share of each person in this annuity will be as under: viz. that of—

$$
\begin{aligned}
A &= 21{\cdot}694 - \tfrac{1}{2}(16{\cdot}748 + 15{\cdot}131) + \tfrac{1}{3} \times 12{\cdot}976 = 10{\cdot}080 \\
B &= 19{\cdot}556 - \tfrac{1}{2}(16{\cdot}748 + 14{\cdot}449) + \tfrac{1}{3} \times 12{\cdot}976 = 8{\cdot}282 \\
C &= 17{\cdot}143 - \tfrac{1}{2}(15{\cdot}131 + 14{\cdot}449) + \tfrac{1}{3} \times 12{\cdot}976 = 6{\cdot}678
\end{aligned}
$$

and the sum of all these respective shares, or 25·040 is the value of the annuity on the longest of the three lives, or the total value of the purchase.

COROLLARY I.[1]

§ 84. If only two lives, A and B, are concerned in the purchase

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[1] Simpson, Prob. 18, and his *Sup.* Prob. 21. Dodson, vol. iii. Ques. 72.

(during whose joint continuance the annuity is to be enjoyed equally between them, but on the decease of either of them it is to belong wholly to the survivor), the value of the share of A will be $a_x-\frac{1}{2}a_{xy}$; and that of B will be $a_y-\frac{1}{2}a_{xy}$: whence the following rule for two lives.

§ 85. ***From the value of an annuity on the given life, subtract half the value of an annuity on the two joint lives: the remainder will be the share of the given life required.***

For examples of the use and application of this corollary, see Question XX. in Chapter XII.

COROLLARY II.

§ 86. On the other hand, let the number of lives concerned in the purchase be ever so great, the share of any given life may be readily determined, provided the annuity be always equally divided among the surviving lives. For, let G denote the value of an annuity on the given life; G_1 the sum of the values of an annuity on all the joint lives arising from combining the given life with each *one* of the others; G_2 the sum of the values of an annuity on all the joint lives arising from combining the given life with each *two* of the others; G_3 the sum of the values of an annuity on all the joint lives arising from combining the given life with each *three* of the others, and so on. Then will $G-\frac{1}{2}G_1+\frac{1}{3}G_2-\frac{1}{4}G_3+$&c. denote the share of the given life, or the value which he ought to contribute towards the purchase.

PROBLEM IX.[1]

§ 87. A, B and C agree to purchase an annuity on the longest of their lives, which is to be divided amongst them in the following manner: A and B are to enjoy it equally during their joint lives; but on the decease of either of them it is to be equally divided between A and C, or B and C, the two remaining persons; and lastly, to be enjoyed wholly by the survivor. To find the value of their respective shares:

SOLUTION.

The expectation of A on what he may happen to receive at the end of any one year, may be considered in three parts, as depending on so many different events: 1. A and B may be both alive, the probability of which at the end of the first year is $\frac{l_x\, l_{y|1}}{l_x\, l_y}$, in which case he will receive $\frac{1}{2}$ the annuity: 2. A and C may be living and B dead, the probability of which at the end of the first year is $\frac{l_x l_{z|1}}{l_x l_z}(1-\frac{l_{y1}}{l_y})$, in which case he will also receive $\frac{1}{2}$ the annuity: 3. A may be the only person living, the probability of which at the end of the first year is $\frac{l_{x1}}{l_x}(1-\frac{l_{y1}}{l_y})\,(1-\frac{l_{z1}}{l_z})$, in which case he will come in for the whole annuity. Now the sum of all these values, being multiplied by v, will give $v.\,(\frac{l_{x1}}{l_x}-\frac{l_x l_{y|1}}{2\,l_x l_y}-\frac{l_x l_{z|1}}{2\,l_x l_z}+\frac{l_x l_y l_{z|1}}{2\,l_x l_y l_z})$ for the total value of the expectation of A, on what he may happen to receive at the end of the first year.

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[1] Dodson, vol. iii. Ques. 75.

By a similar process we may find the value of his expectation on what he may happen to receive at the end of the second, third, and every subsequent year, to the utmost extent of human life. The sum of all which terms, or the series

$$\begin{array}{l} v.\ \left(\frac{l_{x1}}{l_x}-\frac{l_x l_{y|1}}{2\,l_x l_y}-\frac{l_x l_{z|1}}{2\,l_x l_z}+\frac{l_x l_y l_{z|1}}{2\,l_x l_y l_z}\right)+ \\ v^2\ \left(\frac{l_{x2}}{l_x}-\frac{l_x l_{y|2}}{2\,l_x l_y}-\frac{l_x l_{z|2}}{2\,l_x l_z}+\frac{l_x l_y l_{z|2}}{2\,l_x l_y l_z}\right)+ \\ v^3\ \left(\frac{l_{x3}}{l_x}-\frac{l_x l_{y|3}}{2\,l_x l_y}-\frac{l_x l_{z|3}}{2\,l_x l_z}+\frac{l_x l_y l_{z|3}}{2\,l_x l_y l_z}\right)+ \\ \qquad\&c. \qquad\qquad \&c. \qquad\qquad \&c. \end{array}$$

will denote the total value of his interest in the annuity. But the sum of these several perpendicular series is evidently equal to $a_x-\frac{1}{2}a_{x\cdot y}-\frac{1}{2}a_{x\cdot z}+\frac{1}{2}a_{x\cdot y\cdot z}$: whence we have the following rule.

§ 88. *From the value of an annuity on the life* A, *subtract half the sum of the values of an annuity on the joint lives* A B *and* A C; *to the remainder add half the value of an annuity on the three joint lives* A B C: *the sum will be the share which* A *ought to contribute towards the purchase.*

§ 89. With respect to the value of B's interest, it is evident, that his expectation of receiving the annuity in any one year will depend on the same events, *mutatis mutandis*, as that of A: wherefore, by substituting the values, thence arising, in the general expression above given, we shall have $a_y-\frac{1}{2}a_{x\cdot y}-\frac{1}{2}a_{y\cdot z}+\frac{1}{2}a_{x\cdot y\cdot z}$ for the value of B's interest: whence the following rule.

§ 90. *From the value of an annuity on the life* B *subtract half the sum of the values of an annuity on the joint lives* A B *and* B C; *to the remainder add half the value of an annuity on the three joint lives* A B C: *the sum will be the share which* B *ought to contribute towards the purchase.*

§ 91. But with respect to C's interest, it will appear, that his expectation of receiving the annuity in any one year may be considered in the three following points, as depending on so many different events: 1. A and C may be living and B dead, the probability of which at the end of the first year is $\frac{l_x l_{z|1}}{l_x l_z} \times (1 - \frac{l_{y1}}{l_y})$, in which case he will receive $\frac{1}{2}$ the annuity: 2. B and C may be living and A dead, the probability of which, at the end of the first year, is $\frac{l_y l_{z|1}}{l_y l_z} \times (1 - \frac{l_{x1}}{l_x})$, in which case he will receive also $\frac{1}{2}$ the annuity: 3. C may be the only person living, the probability of which, at the end of the first year, is $\frac{l_{z1}}{l_z} (1 - \frac{l_{x1}}{l_x}) (1 - \frac{l_{y1}}{l_y})$, in which case he will receive the whole annuity. The sum of these values, therefore, being multiplied by v will give $v (\frac{l_{z1}}{l_z} - \frac{l_x l_{z|1}}{2 l_x l_z} - \frac{l_y l_{z|1}}{2 l_y l_z})$ for the total value of the expectation of C, on what he may happen to receive at the end of the first year.

By pursuing the same steps, we may find the value of his expectation on what he may happen to receive at the end of the second, third, and every subsequent year to the utmost extent of human life. The sum of all which terms will be the total value of his interest in the annuity, and will be evidently equal to $a_z - \frac{1}{2} a_{x \cdot z} - \frac{1}{2} a_{y \cdot z}$: whence the following rule.

§ 92. *From the value of an annuity on the life* C, *subtract half the sum of the values of an annuity on the joint lives* A C *and* B C; *the remainder will be the share which* C *ought to contribute towards the purchase.*

§ 93. *Example.* Let the three lives be aged 20, 30, and 40, the rate of interest 3 per cent., and the probabilities of living as at *Carlisle.* The value of the share of each person in this annuity will be as under: viz., that of

A = 21·694 − $\frac{1}{2}$ (15·655 + 14·249) + $\frac{1}{2}$ × 12·263 = 12·8735
B = 19·556 − $\frac{1}{2}$ (15·655 + 13·636) + $\frac{1}{2}$ × 12·263 = 11·0420
C = 17·143 − $\frac{1}{2}$ (13·636 + 14·249) . . . . = 3·2005

And the sum of all these values is the value of an annuity on the longest of the three lives.

PROBLEM X.[1]

§ 94. A, B and C agree to purchase an annuity on the longest of their lives, to be divided amongst them in the following manner: A and B are to enjoy it equally during their joint lives; if A dies first, then B and C are to enjoy it equally during their joint lives, and the survivor of them is to have the whole; but if B dies first, then A is to enjoy the whole during his life, and after his decease it is to devolve wholly to C: To find the value of their respective shares.

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[1] Simpson, Prob. 19. Dodson, vol. iii. Ques. 74.

SOLUTION.

The expectation of A, on what he may happen to receive at the end of any one year, may be considered in two points, as depending on so many different events: 1. A and B may be both alive, the probability of which at the end of the first year is $\frac{l_x\, l_{y|1}}{l_x\, l_y}$, in which case he will receive $\frac{1}{2}$ the annuity. 2. A may be living and B dead, the probability of which at the end of the first year is $\frac{l_{x1}}{l_x} \times (1 - \frac{l_{y1}}{l_y})$, in which case he will enjoy the whole annuity. The sum of these two values being multiplied by v, will give $v\,(\frac{l_{x1}}{l_x} - \frac{l_x\, l_{y|1}}{2\, l_x\, l_y})$ for the total value of the expectation of A, on what he may happen to receive at the end of the first year.

By a similar process we may find the value of his expectation on what he may happen to receive at the end of the second, third, and every subsequent year, to the utmost extent of human life. The sum of all which terms will be the total value of his interest in the annuity, and will be evidently equal to $a_x - \frac{1}{2}\, a_{x \cdot y}$: whence we have the following rule.

§ 95. *From the value of an annuity on the life* A, *subtract half the value of an annuity on the joint lives* A B: *the remainder will be the value which* A *ought to contribute towards the purchase.*

§ 96. In order to determine the share of B, it should be observed, that his expectation on what he may happen to receive at the end of any one year may be considered in three points, as depending on so many different events: 1. A and B may be both alive, the probability of which at the end of the first year is $\frac{l_x\, l_{y|1}}{l_x\, l_y}$, in which case he will receive $\frac{1}{2}$ the annuity: 2. B and C may be alive and A dead, the probability of which at the end of the first year is $\frac{l_y\, l_{z|1}}{l_y\, l_z} \times (1 - \frac{l_{x1}}{l_x})$, in which case also he will receive $\frac{1}{2}$ the annuity: 3. B may be the only person living, the probability of which at the end of the first year is $\frac{l_{y1}}{l_y} \times (1 - \frac{l_{x1}}{l_x}) \times (1 - \frac{l_{z1}}{l_z})$, in which case he will enjoy the whole annuity. And the sum of all these values, being multiplied by v, will give

$$v\left(\frac{l_{y1}}{l_y}-\frac{l_x\,l_{y|1}}{2\,l_x\,l_y}-\frac{l_y\,l_{z|1}}{2\,l_y\,l_z}+\frac{l_x\,l_y\,l_{z|1}}{2\,l_x\,l_y\,l_z}\right)$$

for the total value of the expectation of B, on what he may happen to receive at the end of the first year.

By a similar process we may find the value of his expectation for the second, third, and every subsequent year, to the utmost extent of human life. The sum of all which terms will be the total value of his interest in the annuity; and will be evidently equal to $a_y-\frac{1}{2}a_{x\cdot y}-\frac{1}{2}a_{y\cdot z}+\frac{1}{2}a_{x\cdot y\cdot z}$: whence the following rule.

§ 97. *From the value of an annuity on the life* B, *subtract half the sum of the values of an annuity on the joint lives* A B, *and* B C; *to the remainder add half the value of an annuity on the three joint lives* A B C: *the sum will be the value which* B *ought to contribute towards the purchase.*[1]

§ 98. It remains now to determine the value of the share of C, whose expectation of receiving the annuity in any one year may be considered in two points: 1. B and C may be both alive and A dead, the probability of which at the end of the first year is $\frac{l_y\,l_{z|1}}{l_y\,l_z}\times(1-\frac{l_{x1}}{l_x})$, in which case he will receive ½ the annuity: 2. C only may be living, the probability of which at the end of the first year is $\frac{l_{z1}}{l_z}\times(1-\frac{l_{x1}}{l_x}\times(1-\frac{l_{y1}}{l_y})$, in which case he will enjoy the whole annuity. And the sum of these two values, being multiplied by v, will give $v(\frac{l_{z1}}{l_z}-\frac{l_x\,l_{z|1}}{l_x\,l_z}-\frac{l_y\,l_{z|1}}{2\,l_y\,l_z}+\frac{l_x\,l_y\,l_{z|1}}{2\,l_x\,l_y\,l_z})$ for the total value of the expectation of C, on what he may happen to receive at the end of the first year.

By a similar method of proceeding we may find the value of his expectation for the second, third, and every subsequent year, to the utmost extent of human life. The sum of all which terms will express the total value of his interest in the annuity, and will be evidently equal to $a_z-a_{x\cdot z}-\frac{1}{2}a_{y\cdot z}+\frac{1}{2}a_{x\cdot y\cdot z}$: whence the following rule.

§ 99. *From the value of an annuity on the life* C, *subtract the value of an annuity on the joint lives* A C, *and also half the value of an annuity on the joint lives* B C; *to the remainder add half the value of an annuity on the joint lives* A B C: *the sum will be the share which* C *ought to contribute towards the purchase.*

§ 100. *Example.* Suppose the three lives to be aged 20, 30, and 40;

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[1] The value of B's interest in this annuity is the same as that in the preceding problem.

the rate of interest 3 per cent., and the probabilities of living as at *Carlisle.* The value of the share of each person in this annuity will be as under: viz. that of

A = 21·694 − $\frac{1}{2}$ × 15·655 . . . . = 13·8165
B = 19.556 − $\frac{1}{2}$ (15·655 + 13·636) + $\frac{1}{2}$ × 12·263 = 11·0420
C = 17·143 − 14·249 − $\frac{1}{2}$ × 13·636 + $\frac{1}{2}$ × 12·263 = 2·8940

and the sum of these three values is the value of an annuity on the longest of the three lives.

PROBLEM XI.[1]

§ 101. A, B and C agree to purchase an annuity on the longest of their lives, to be divided amongst them in the following manner: A is to enjoy the whole annuity during his life; but after his decease it is to be divided equally between B and C during their joint lives; and the survivor of them is to have the whole: To find the value of their respective shares.

SOLUTION.

The value of A's interest in this annuity, or the share which he ought to contribute towards the purchase, is evidently equal to the value of an annuity on his life: that is, equal to a_x.

§ 102. As to the share of B, the expectation on what he may happen to receive at the end of any one year may be considered in two points, as depending on so many different events: 1. B and C may be alive and A dead, the probability of which at the end of the first year is $\frac{l_y\, l_{z|1}}{l_y\, l_z} \times (1 - \frac{l_{x1}}{l_x})$, in which case he will receive $\frac{1}{2}$ the annuity: 2. B may be the only person living, the probability of which at the end of the first year is $\frac{l_{y1}}{l_y} \times (1 - \frac{l_{x1}}{l_x}) \times (1 - \frac{l_{z1}}{l_z})$, in which case he will enjoy the whole annuity. And the sum of these two values, being multiplied by v, will give

$$v\left(\frac{l_{y1}}{l_y} - \frac{l_x\, l_{y|1}}{l_x\, l_y} - \frac{l_y\, l_{z|1}}{2\, l_y\, l_z} + \frac{l_x\, l_y\, l_{z|1}}{2\, l_x\, l_y\, l_z}\right)$$

for the total value of the expectation of B, on what he may happen to receive at the end of the first year.

By a similar method of proceeding we may find the value of his expectation for the second, third, and every subsequent year, to the utmost extent of human life. The sum of all which terms will be the total value of his interest in the annuity, and will be found equal to $a_y - a_{xy} - \frac{1}{2}a_{yz} + \frac{1}{2}a_{xyz}$: whence the following rule.

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[1] Simpson's *Sup*. Prob. 22. Dodson, vol. iii. Ques. 76.

§ 103. *From the value of an annuity on the life* B, *subtract the value of an annuity on the joint lives* A B, *and also half the value of an annuity on the joint lives* B C; *to the remainder add half the value of an annuity on the joint lives* A B C: *the sum will be the value which* B *ought to contribute towards the purchase.*

§ 104. With respect to the share of C, it is evident that his expectation of receiving the annuity at the end of any one year will depend on the same events, ***mutatis mutandis***, as that of B. Therefore, by pursuing the same steps, we shall find that the value of his interest in the annuity will be equal to $a_z - a_{xz} - \frac{1}{2}a_{yz} - \frac{1}{2}a_{xyz}$: whence we have the following rule.

§ 105. *From the value of an annuity on the life* C, *subtract the value of an annuity on the joint lives* A C, *and also half the value of an annuity on the joint lives* B C; *to the remainder add half the value of an annuity on the joint lives* A B C: *the sum will be the value which* C *ought to contribute towards the purchase.*[1]

§ 106. *Example.* Let the three lives be aged 20, 30, and 40, the rate of interest 3 per cent., and the probabilities of living as at *Carlisle*. The value of the share of each person in this annuity will be as under: viz., that of

$A = 21{\cdot}694 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad = 21{\cdot}694$

$B = 19{\cdot}556 - 15{\cdot}655 - \frac{1}{2} \times 13{\cdot}636 + \frac{1}{2} \times 12{\cdot}263 = \;\; 3{\cdot}214$

$C = 17{\cdot}143 - 14{\cdot}249 - \frac{1}{2} \times 13{\cdot}636 + \frac{1}{2} \times 12{\cdot}263 = \;\; 2{\cdot}207$

and the sum of these three values is the value of an annuity on the longest of the three lives.

PROBLEM XII.

§ 107. A, B and C agree to purchase an annuity on the longest of their lives, to be enjoyed wholly by each of them in succession: that is, A is to enjoy it first, for his life; at his decease, B is to enter upon it next; and C last: To find the value of their respective shares.

SOLUTION.

The value of A's interest in this annuity, or the share which he ought to contribute towards the purchase, is evidently equal to the value of an annuity on his life that is equal to a_x.

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[1] The value of C's interest in this annuity is the same as that in the preceding problem.

§ 108. The value of B's interest, or the share which he ought to contribute towards the purchase, is equal to the value of a reversionary annuity on his life after the decease of A: that is (by the Scholium in page 53), equal to $a_y - a_{xy}$.

§ 109. The value of C's interest, or the share which he ought to contribute towards the purchase, is equal to the value of a reversionary annuity on his life after the extinction of the longest of the two lives A and B: that is (by the same Scholium), equal to $a_z - a_{xz} - a_{yz} + a_{xyz}$.

§ 110. *Example.* Suppose the three lives to be aged 20, 30, and 40; the rate of interest 3 per cent., and the probabilities of living as at *Carlisle.* The value of the share of each person in this annuity will be as under: viz. that of—

$$\begin{aligned} A &= 21{\cdot}694 \quad \ldots\ldots\ldots\ldots &&= 21{\cdot}694 \\ B &= 19{\cdot}556 - 16{\cdot}748 \quad \ldots\ldots\ldots &&= 2{\cdot}808 \\ C &= 17{\cdot}143 - (15{\cdot}131 + 14{\cdot}449) + 12{\cdot}976 &&= 0{\cdot}539 \end{aligned}$$

and the sum of these three values will be the value of an annuity on the longest of the three lives.

COROLLARY.

§ 111. If only two lives, A and B, are concerned in the purchase, the value of their respective shares will be the same as above given.

PROBLEM XIII.

§ 112. A, B, and C agree to purchase an annuity, to continue as long as any two of their lives are in being; and which is to be equally divided between them whilst they are all living; but on the decease of either of them it is to be equally divided between the two survivors during their joint lives: To find the value of their respective shares.

SOLUTION.

The expectation of A, on what he may happen to receive at the end of any one year, may be considered in three parts, as depending on so many different events:—1. A, B, and C may be all living, in which case he will receive $\frac{1}{3}$ of the annuity:—2. A and B may be living and C dead, in which case he will receive $\frac{1}{2}$ the annuity:—3. A and C may be living and B dead, in which case he will receive also $\frac{1}{2}$ the annuity. Therefore the sum of these expectations for the first, second, third, &c., years, continued to the utmost extent of human life, will be the total value of A's share in the annuity. But the value of these several expectations has been already found in the solution to Prob. VIII.; they being precisely the same as the first three which are there mentioned: and the sum of which, for every year of human life, will be found equal to $\frac{1}{2}a_{xy}+\frac{1}{2}a_{xz}-\frac{2}{3}a_{xyz}$.

§ 113. With respect to the share of B or C, their expectation of receiving the annuity in any one year will depend on the same events, *mutatis mutandis*, as that of A: whence it follows that by substituting the values, thence arising, in the general expression above given, we shall have $\frac{1}{2}a_{xy}+\frac{1}{2}a_{yz}-\frac{2}{3}a_{xyz}$ for the value of B's interest; and $\frac{1}{2}a_{xz}+\frac{1}{2}a_{yz}-\frac{2}{3}a_{xyz}$ for the value of C's interest in the given annuity: whence the following rule for determining the value of the share of any given life in the annuity.

§ 114. *Subtract two-thirds of the value of an annuity on the three joint lives, from half the sum of the values of an annuity on both the joint lives arising from the combination of the given life with each one of the others: the remainder will be the value of the share of such given life.*

§ 115. *Example.* Suppose the three lives to be aged 20, 30, and 40; interest 3 per cent., and the probabilities of living as at *Carlisle.* The value of the share of each person will be as under: viz. that of—

$$\begin{aligned}
A &= \tfrac{1}{2}\,(16{\cdot}748 + 15{\cdot}131) - \tfrac{2}{3} \times 12{\cdot}976 = 7{\cdot}288 \\
B &= \tfrac{1}{2}\,(16{\cdot}748 + 14{\cdot}449) - \tfrac{2}{3} \times 12{\cdot}976 = 6{\cdot}948 \\
C &= \tfrac{1}{2}\,(15{\cdot}131 + 14{\cdot}449) - \tfrac{2}{3} \times 12{\cdot}976 = \underline{6{\cdot}139}
\end{aligned}$$

and the sum of these three values, or 20·375

is the value of an annuity on the three lives, to continue as long as any two of them are in being together.

PROBLEM XIV.

§ 116. A, B, and C agree to purchase an annuity, to continue as long as any two of their lives are in being; and which is to be divided amongst them in the following manner:—A and B are to enjoy it equally during their joint lives; but on the death of either of them, it is to be equally divided between the two survivors, for their joint lives: To find their respective shares.

SOLUTION.

The expectation of A on what he may happen to receive at the end of any one year may be considered in two parts, as depending on so many different events: 1. A and B may be both alive, in which case he will receive ½ the annuity: 2. A and C may be living and B dead, in which case he will receive $\frac{1}{2}$ the annuity. Therefore the sum of these expectations for the first, second, third, &c., years, to the utmost extent of human life, will be the total value of A's share in the annuity. But the value of these several expectations has been already found in the solution to Prob. IX.; they being precisely the same as the first two there mentioned: and the sum of which, for every year of human life, will be found equal to $\frac{1}{2}a_{xy}+\frac{1}{2}a_{xz}-\frac{1}{2}a_{xyz}$: whence the following rule.

§ 117. *Subtract the value of an annuity on the joint lives* A B C, *from the sum of the values of an annuity on each of the joint lives* A B *and* A C: *half the remainder will be the value which* A *ought to contribute towards the purchase.*

§ 118. With respect to the share of B, it is evident that his expectation of receiving the annuity at the end of any one year will depend on the same events, *mutatis mutandis*, as that of A: wherefore by substituting the values, thence arising, in the general expression above given, we shall have $\frac{1}{2}a_{xy}+\frac{1}{2}a_{xz}-\frac{1}{2}a_{xyz}$ for the value of B's interest in the annuity: whence the following rule.

§ 119. *Subtract the value of an annuity on the joint lives* A B C, *from the sum of the values of an annuity on each of the joint lives* A B *and* B C: *half the remainder will be the value which* B *ought to contribute towards the purchase.*

§ 120. But with respect to the share of C, it will appear that his expectation of receiving the annuity at the end of any one year may be considered under the two following points, as depending on so many different events: 1. A and C may be living and B dead, in which case he will receive $\frac{1}{2}$ the annuity: 2. B and C may be living and A dead, in which case also he will receive $\frac{1}{2}$ the annuity. Therefore the sum of these two expectations, for the first, second, third, &c., years, to the utmost extent of human life, will be the total value of C's interest in the annuity. But the value of these expectations has been already found in the solution to Prob. IX.; they being precisely the same as the first two which are mentioned in the investigation of C's share in that annuity: and the sum of which, for every year of human life, will be found equal to $\frac{1}{2}a_{xz}+\frac{1}{2}a_{yz}-a_{xyz}$: whence the following rule.

§ 121. *Subtract the value of an annuity on the joint lives* A B C, *from half the sum of the values of an annuity on each of the joint lives* A C *and* B C: *the difference will be the share which* C *ought to contribute towards the purchase.*

§ 122. *Example.* Suppose the three lives to be aged 20, 30, and 40; the rate of interest 3 per cent., and the probabilities of living as at *Carlisle.* The value of the share of each person in this annuity will be as under: viz. that of

$$A = \tfrac{1}{2}\,(16{\cdot}748 + 15{\cdot}131 - 12{\cdot}976) = 9{\cdot}4515$$
$$B = \tfrac{1}{2}\,(16{\cdot}748 + 14{\cdot}449 - 12{\cdot}976) = 9{\cdot}1105$$
$$C = \tfrac{1}{2}\,(15{\cdot}131 + 14{\cdot}449 - 12{\cdot}976) = 8{\cdot}4020$$

and the sum of these three values is the total value of the annuity on the three lives, to continue as long as any two of them are in being together.

PROBLEM XV.

§ 123. A, B, and C agree to purchase an annuity to continue as long as any two of them are in being together; and which is to be divided between them in the following manner: A and B are to enjoy it equally during their joint lives; if A dies first, then B and C are to enjoy it equally during their joint lives; but if B dies first, then A is to enjoy the whole during the joint lives of A and C: To find the value of their respective shares.

SOLUTION.

The expectation of A on what he may happen to receive at the end of any one year may be considered in two parts, as depending on so many different events:—1. A and B may be both alive, the probability of which at the end of the first year is $\frac{l_{x1}\,l_{y1}}{l_x\,l_y}$, in which case he will receive $\frac{1}{2}$ the the annuity:—2. A and C may be living and B dead, the probability of which at the end of the first year is $\frac{l_{x1}\,l_{z1}}{l_x\,l_z}\,(1-\frac{l_{y1}}{l_y})$, in which case he will come in for the whole annuity. The sum of these two values multiplied by v, will give $v\left(\frac{l_{x1}\,l_{y1}}{2l_x\,l_y}+\frac{l_{x1}\,l_{z1}}{l_x\,l_z}-\frac{l_{x1}\,l_{y1}\,l_{z1}}{l_x\,l_y\,l_z}\right)$ for the total value of the expectation of A on what he may happen to receive at the end of the first year.

By a similar process we may find the value of his expectation for the second, third, and every subsequent year to the utmost extent of human life: the sum of all which terms will be the total value of his interest in the annuity; and will be found equal to $\frac{1}{2}a_{xy}+\frac{1}{2}a_{xz}-a_{xyz}$: whence the following rule.

§ 124. *Add half the value of an annuity on the joint lives* A B, *to the value of an annuity on the joint lives* A C; *from the sum, subtract the value of an annuity on the joint lives* A B C; *the remainder will be the value of the share of* A.

§ 125. The expectation of B on what he may happen to receive at the end of any one year may also be considered in two parts as depending on so many different events :—1. A and B may be both alive, the probability of which at the end of the first year is $\frac{l_{x1}\,l_{y1}}{l_x\,l_y}$, in which case he will receive $\frac{1}{2}$ the annuity :—2. B and C may be alive and A dead, the probability of which at the end of the first year is $\frac{l_{y1}\,l_{z1}}{l_y\,l_z}(1-\frac{l_{x1}}{l_x})$, in which case, likewise, he will receive $\frac{1}{2}$ the annuity. The sum of these two values therefore being multiplied by v, will give $v\left(\frac{l_{x1}\,l_{y1}}{2l_x\,l_y}+\frac{l_{y1}\,l_{z1}}{2l_y\,l_z}-\frac{l_{x1}\,l_{y1}\,l_{z1}}{2l_x\,l_y\,l_z}\right)$ for the expectation of B on what he may happen to receive at the end of the first year.

By a similar process we may find the value of his expectation on what he may happen to receive at the end of the second, third, and every subsequent year, to the utmost extent of human life : the sum of all which terms will be the total value of his interest in the annuity ; and will be found equal to $\frac{1}{2}(a_{xy}+a_{yz}-a_{xyz})$: whence the following rule.

§ 126. *Subtract the value of an annuity on the three joint lives* A B C, *from the sum of the values of an annuity on each of the joint lives* A B *and* B C : *half the remainder will be the value of the share of* B.[1]

§ 127. With respect to the share of C, it is solely equal to half the value of a reversionary annuity on the joint lives B C, after the life of A ; which reversionary annuity is by the Scholium in page 53, expressed by $\frac{1}{2}(a_{yz}-a_{xyz})$: whence the following rule.

§ 128. *Subtract the value of an annuity on the joint lives* A B C, *from the value of an annuity on the joint lives* B C : *half the remainder will be the value of the share of* C.

§ 129. *Example.* Suppose the three lives to be aged 20, 30, and 40 ; interest at 3 per cent., and the probabilities of living as at *Carlisle.* The value of the share of each person will be as under, viz., that of

$$\begin{aligned}
A &= \tfrac{1}{2} \times 16{\cdot}748 + 15{\cdot}131 - 12{\cdot}976 &&= 10{\cdot}5290\\
B &= \tfrac{1}{2}\ (16{\cdot}748 + 14{\cdot}449 - 12{\cdot}976) &&= 9{\cdot}1105\\
C &= \tfrac{1}{2}\ (14{\cdot}449 - 12{\cdot}976)\ \ldots &&= 0{\cdot}7365
\end{aligned}$$

and the sum of all these three values will be the total value given for the purchase.

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[1] The value of B's interest in this annuity is the same as that in the preceding Problem.

PROBLEM XVI.[1]

§ 130. D and his heirs, as soon as any *two* of three given lives A, B, C, become extinct, are to enter upon an annuity in order to enjoy the same during the life of the survivor: To find the value of his interest therein.

SOLUTION.

This annuity is evidently to continue during the longest of the three given lives; and the value of the same without any restriction would be (by Prob. II.) equal to $a_x+a_y+a_z-a_{xy}-a_{xz}-a_{yz}+a_{xyz}$: but, since D (or his heirs) is not to receive anything during the continuance of any two of these lives, the value of an annuity depending on this contingency must be subtracted from the preceding expression; and which value is (by Prob. III.) denoted by $a_{xy}+a_{xz}+a_{yz}-2a_{xyz}$. Therefore $a_x+a_y+a_z-2a_{xy}-2a_{xz}-2a_{yz}+3a_{xyz}$ will be the value required: whence the following rule.

§ 131. *From the sum of the values of an annuity on each of the single lives, subtract twice the sum of the values of an annuity on each pair of joint lives; to the remainder add thrice the value of an annuity on the three joint lives: the sum will be the interest of D in this annuity, or the value of the reversion required.*

§ 132. *Example.* An estate is held on three lives whose ages are 20, 30, and 40, the income of which, as soon as any two of these lives become extinct, is to belong to D and his heirs during the continuance of the third life: what is the interest of D in this lease, reckoning the probabilities of living as at *Carlisle*, and the rate of interest 3 per cent.?

The sum of the values of an annuity on each of the single lives is $21{\cdot}694+19{\cdot}556+17{\cdot}143=58{\cdot}393$; the sum of the values of an annuity on each pair of joint lives is $16{\cdot}748+15{\cdot}131+14{\cdot}449=46{\cdot}328$; and the value of an annuity on the three joint lives is 12·976. Therefore $58{\cdot}393-2\times46{\cdot}328+3\times12{\cdot}976=4{\cdot}665$ will be the value required.

PROBLEM XVII.[2]

§ 133. D and his heirs, as soon as any *one* of three given lives, A, B, C, becomes extinct, are to enter upon an annuity in order to enjoy the same during the continuance of the longest of the remaining lives. To find the value of his interest therein.

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[1] Simpson's *Sup.* Prob. 20. Dodson, vol. iii. Ques. 79.

[2] Dodson, vol. iii. Ques. 78.

SOLUTION.

It is evident, in this case also, that the annuity is to continue during the longest of the three given lives; and such would be the value of it to D (or his heirs) were he to enter upon it immediately: but since he is not to receive anything during the continuance of all the joint lives, the value of an annuity on those joint lives must be subtracted from the value of an annuity on the longest of the three lives, in order to obtain the required value. Wherefore, $a_x+a_y+a_z-a_{xy}-a_{xz}-a_{yz}+a_{xyz}-a_{xyz}=a_x+a_y+a_z-a_{xy}-a_{xz}-a_{yz}$ will be the value of his interest in the annuity: whence the following rule:—

§ 134. *From the sum of the values of an annuity on each of the single lives, subtract the sum of the values of an annuity on each pair of joint lives: the remainder will be the interest of D in this annuity, or the value of the reversion required.*

§ 135. *Example.* Suppose the lives to be aged 20, 30, and 40; the rate of interest 3 per cent., and the probabilities of living as at *Carlisle.* In which case the value required would be

$$21{\cdot}694+19{\cdot}556+17{\cdot}143-(16{\cdot}748+15{\cdot}131+14{\cdot}449)=12{\cdot}065.$$

COROLLARY.

§ 136. Had two lives only, A and B, been concerned in this question, the value of the interest of D in the annuity would (by the same method of reasoning) become equal to $a_x+a_y-2a_{xy}$: whence the following rule.

§ 137. *From the sum of the values of an annuity on each single life, subtract twice the value of an annuity on the two joint lives: the remainder will be the value required.*

For examples of the use and application of this corollary, see Question XXI. in Chapter XII.

PROBLEM XVIII.[1]

§ 138. D and his heirs, as soon as any *one* of three given lives, A, B, C, becomes extinct, are to enter upon an annuity in order to enjoy the same during the joint continuance of the survivors: To find the value of his interest therein.

SOLUTION.

In this case, the annuity is to continue only as long as any two of the given lives are in being; and such would be the value of it to D, were he to enter on it immediately: but since he is to receive nothing during the joint continuance of all the lives, the value of an annuity on those joint lives must be subtracted from the value of an annuity to continue as long as any two of them are in being together. Therefore $a_{xy}+a_{xz}+a_{yz}-2a_{xyz}-a_{xyz}=a_{xy}+a_{xz}+a_{yz}-3a_{xyz}$ will be the value required: whence the following rule.

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[1] Dodson, vol. iii. Ques. 80.

§ 139. *From the sum of the values of an annuity on each pair of joint lives, subtract thrice the value of an annuity on the three joint lives: the remainder will be the value required.*

§ 140. *Example.* Suppose the lives to be aged 20, 30, and 40; the rate of interest 3 per cent., and the probabilities of living as at *Carlisle.* In which case the required value would be

$$16{\cdot}748+15{\cdot}131+14{\cdot}449-3\times 12{\cdot}976=7{\cdot}400.$$

SCHOLIUM.

§ 141. The preceding problems contain some of the most general cases of survivorship, where not more than three lives are concerned; they are by no means offered as the whole which may arise, since the conditions of the problems are capable of infinite variety: but, by attention to the method of deducing the values in the cases here given, it is presumed that no difficulty will be found in investigating any other case which may present itself.

§ 142. I would here observe, however, that the values, in all these problems, are deduced from the *whole continuance* of the lives on which the annuities depend: but if, from the conditions of the problem, the contingencies are *Deferred* or *Temporary*, then the annuities which result therefrom will be affected in a similar manner. In such cases, therefore, we must substitute the value of such deferred or temporary annuities, instead of annuities on the whole continuance of the lives, and proceed with such substituted values in the same manner as with the whole values of the annuities: agreeably to what I have already mentioned in Prob. II. cor. 2 and 4: and cor. 1 to the Scholium in page 53.

CHAPTER V.

ON REVERSIONARY ANNUITIES DEPENDING UPON A PARTICULAR ORDER OF SURVIVORSHIP.

§ 143. Among the cases of reversionary annuities, involving not more than three lives, and which are given in the Scholium at the end of the third Chapter (see page 54), there are two which oftentimes involve a contingency for which it is extremely difficult to find a correct algebraical solution fit for general use. These are the second and fourth cases there mentioned; and the contingency alluded to is, that one of the two lives *in particular*, P or Q, shall die before or after the other.

As the investigations necessary to the solution of these cases will be of considerable importance in the further progress of this work, I shall devote a separate chapter to the examination of the subject. The solution of the cases themselves will lead to that of some other questions in Survivorships, which for want thereof could not be inserted in the preceding chapter.

§ 144. Instead of the lives A, P, Q, as given in the Scholium above mentioned, let the lives be A, B, C; of which the lives B and C are supposed to be *in possession*, and the life A *in reversion*. The value of a reversionary annuity on the life A after the joint lives B C, is equal to $a_x - a_{xyz}$: but if A is not to enjoy the annuity unless B be the *first* that dies of the two joint lives B C, it is evident that the value of the annuity will be considerably affected by this restriction.

In like manner, the value of a reversionary annuity on the life A after the longest of two lives B and C, is equal to $a_x - a_{xy} - a_{xz} + a_{xyz}$: but if A is not to enjoy the annuity unless B be the *last* that dies of the two lives B C, it is evident the value of an annuity will in this case also be considerably affected by this restriction.

When the two lives in possession are of the same age, or nearly so, it will be an *equal* chance which of them dies before or after the other: but as this is not always the case, it would be extremely desirable to obtain an expression fit for general use, which should express this probability *for every year of human life;* because the expectation of receiving *each* year's rent will depend on such contingency. Unfortunately however, this probability cannot be denoted by a *constant* quantity till after the extinction of the oldest of the two lives; because the chance of survivorship varies in every succeeding year till that period, and even then we can only approximate to the true value. When the difference between the ages of the lives in possession is not very great, the error will be but trifling (as I have just observed) by supposing it to be an *equal* chance which of those lives dies before or after the other, during the probable time of their joint continuance; but when there is any considerable difference between the ages of the two lives, this cannot be an accurate method of proceeding. *After* that period, however, the contingency may in all cases be expressed with a tolerable degree of correctness. But previous to the investigation of any of the problems connected with this subject, it will be necessary to premise the following lemmata.

LEMMA I.[1]

§ 145. **To determine the probability that out of two given lives A and B, one of them in particular, A, shall fail *before* the other.**

SOLUTION.

It is evident that this circumstance may take place in any one year (1.) by A dying *in* that year and B surviving it; (2.) by the extinction of both the lives *in* the year, A having died first. The probability of the first event happening in the first year is, by §§ 24 and 25, $\frac{d_x}{l_x}\times\frac{l_{y1}}{l_y}$;[2] and the probability of the second event happening in the same period is $\frac{d_x}{l_x}\times\frac{d_y}{l_y}\times\frac{1}{2}$;[3] the sum of these two, therefore, or $\frac{d_x(l_y+l_{y1})}{2l_x\,l_y}$,[4] will denote the whole probability that A dies the first of the two lives in the *first* year.

In like manner, the probability of the first event happening in the second year is $\frac{d_{x1}\times l_{y2}}{l_x\,l_y}$; and the probability of the second event happening in the same period is $\frac{d_{x1}\times d_{y1}}{2l_x\,l_y}$; therefore the sum of these two, or $\frac{d_{x1}(l_{y1}+l_{y2})}{2l_x\,l_y}$, will denote the whole probability that A dies the first of the two lives in the *second* year. And if this value be added to the one just found for the first year, it will give $\frac{d_x(l_y+l_{y1})}{2l_x\,l_y}+\frac{d_{x1}(l_{y1}+l_{y2})}{2l_x\,l_y}$ for the probability that A dies the first of the two lives in *two* years.

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[1] *Phil. Trans.* 1788, Part II. p. 332. Dodson, vol. iii. Ques. 21. Simpson, p. 100. De Moivre, Prob. 18.

[2] In lieu of the character *d*, here introduced to signify the number *dying*, or the decrement of the number living, the Author throughout uses by his notation $a-a'$, $a'-a''$, $a''-a'''$; thus representing the number living at age x minus the number living at age x_1. The former mode was considered more simple, and was therefore adopted.—EDITOR.

[3] In all these annual contingencies I have supposed it to be an *equal* chance that A dies before B, whatever may be the difference between the two ages; and though this is not strictly correct, except in those cases where the two lives are equal, yet as such chance is in the present investigation confined to *each particular year*, and, when applied practically, is involved with other quantities which very much diminish the error that would otherwise arise from this hypothesis, it would be needless to render the solution more complicated and intricate by aiming at greater exactness. It will, however, appear from this remark that the result here obtained is only an *approximation* to the true value, which is more and more correct the further we continue the series. When A is the youngest of the two lives, the fraction $\frac{1}{2}$ will exceed the chance that A dies before B; and consequently the value of p, as found by the lemma, will *exceed* the true value of the probability that A dies before B, in any part of their co-existence. On the other hand, when A is the oldest of the two lives, the chance that A dies before B will be more than $\frac{1}{2}$; and consequently p will in this case *fall short* of the true value of the probability that A dies before B in such period. Proper allowance may be made in such cases when there is any material difference in the ages of the two lives.

[4] The numerator in this fraction is deduced from $2d_x\,l_{y1}+d_x\,d_y$ which would be the original result of the two values, by reason of $l_{y1}+d_y=l_y$.—EDITOR.

By a similar method of proceeding it will be found that $\frac{d_{x2}(l_{y2}+l_{y3})}{2l_x\,l_y}$ will denote the whole probability that A dies the first of the two lives in the *third* year: which being added to the value just found for two years, will give $\frac{d_x(l_y+l_{y1})}{2l_x\,l_y}+\frac{d_{x1}(l_{y1}+l_{y2})}{2l_x\,l_y}+\frac{d_{x2}(l_{y2}+l_{y3})}{2l_x\,l_y}$ for the probability that this event will happen in *three* years.

And so on with respect to the values for all the subsequent years to the utmost extent of human life: and if we make m equal to the difference between the age of the oldest of the given lives and that age in the table of observations at which human life becomes extinct, we shall find that the series

$$\frac{d_x(l_y+l_{y1})}{2l_x\,l_y}+\frac{d_{x1}(l_{y1}+l_{y2})}{2l_x\,l_y}+\frac{d_{x2}(l_{y2}+l_{y3})}{2l_x\,l_y}\;\ldots\ldots+\frac{d_{x+m-1}(l_{y+m-1}+l_{y+m})}{2l_x\,l_y}$$

will express the whole chance of A dying before B, during that period. But this series, reduced to more simple terms, is equal to

$$\frac{1}{2l_x\,l_y}[d_x(l_y+l_{y1})+d_{x1}(l_{y1}+l_{y2})+d_{x2}(l_{y2}+l_{y3})\ldots+d_{x+m-1}(l_{y+m-1}+l_{y+m})],$$

the sum of which I shall denote by p; therefore p will express the probability that A will die before B during the probable term of their joint continuance: and its application will be shown in the solution of the subsequent problems.

§ 146. In finding the numerical value of this series for lives of different ages, but whose common difference of age is the same, it will be most convenient to begin with the last term of it, or $d_{x+m-1}(l_{y+m-1}+l_{y+m})$, and to proceed upwards till we come to the ages of the given lives, as in finding the numerical value of Annuities: see Prob. I. cor. 2. By this method, the process for finding the probability of survivorship between two other lives, whose ages have the same common difference as the given lives, will be considerably abridged; since the latter terms of the series will be common to both cases.

§ 147. *Example.* Let it be required to find the probability that A the youngest of two lives, whose difference of age is 10 years, shall die before B the eldest; according to the probabilities of living as observed in *Carlisle*, amongst males and females *collectively*. The process for finding such values, for every year of human life, will best appear from the following specimen :—

x.	y.	CARLISLE RATE OF MORTALITY.
94	104	$\frac{1}{2\times 40} \times 10 = {\cdot}1250$
93	103	$\frac{1}{2\times 54\times 3}[14(3+1)+10] = {\cdot}2037$
92	102	$\frac{1}{2\times 75\times 5}[21(5+3)+14(3+1)+10] = {\cdot}3120$
91	101	$\frac{1}{2\times 105\times 7}[30(7+5)+21(5+3)+14(3+1)+10] = {\cdot}4040$
90	100	$\frac{1}{2\times 142\times 9}[37(9+7)+30(7+5)+21(5+3)+14(3+1)+10] = {\cdot}4640$
89	99	$\frac{1}{2\times 181\times 11}[39(11+9)+37(9+7)+30(7+5)+21(5+3)+14(3+1)+10] = {\cdot}4937$
88	98	$\frac{1}{2\times 232\times 14}[51(14+11)+39(11+9)+37(9+7)+30(7+5)+21(5+3)+\dots = {\cdot}4989$
87	97	$\frac{1}{2\times 296\times 18}[64(18+14)+51(14+11)+39(11+9)+37(9+7)+30(7+5)+\dots = {\cdot}4963$
86	96	$\frac{1}{2\times 367\times 23}[71(23+18)+64(18+14)+51(14+11)+39(11+9)+37(9+7)+\dots + {\cdot}4857$
85	95	$\frac{1}{2\times 445\times 30}[78(30+23)+71(23+18)+64(18+14)+51(14+11)+39(11+9)+\dots = {\cdot}4619$

By these examples it will be seen that the probability that a life aged 85 will die before another life aged 95, is expressed by the fraction ·4619; where *certainty* is denoted by unity; and that if the two lives were 87 and 97, the probability would be expressed by ·4963. It will be seen also, on inspection, that each year's series is for the most part composed of the terms which form the preceding series: and consequently, that the finding the probability of survivorship between lives of *several* ages, whose common difference is the same, is not much more laborious than finding it for the youngest of those lives.

§ 148. The above example relates to the probabilities of survivorships between the two lives as observed in *Sweden* amongst mankind *at large:* but if one of those lives be a *male* and the other a *female*, the results will be materially different, as may be seen by the two following tables, where the answer will vary according as the male or the female is the oldest of the two lives:—

Age of A (Male).	Age of B (Female).	Probability that A dies first.
87	97	·1209
86	96	·1956
85	95	·2405
84	94	·2811
83	93	·3147
82	92	·3342
81	91	·3478
80	90	·3534

Age of A (Female).	Age of B (Male).	Probability that A dies first.
87	97	·0000
86	96	·0000
85	95	·1228
84	94	·1798
83	93	·2045
82	92	·2437
81	91	·2678
80	20	·2797

Two other tables likewise might be formed for the same ages, viz., for those cases where *both* the lives are males, and for those cases where *both* the lives are females; and the results would in these cases also be different from those above adduced. But enough has been said to enable the reader to calculate the true probability of survivorships according to any case that may come before him.[1]

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[1] Mr. Morgan has given a "Table showing the probabilities of survivorship between two persons of all ages, whose common difference of age is not less than 10, computed from the *Northampton* Table of Observations."—See *Phil. Trans.* vol. lxxviii. p. 337; or Dr. Price's *Obs. on Rev. Pay.* vol. i. p. 406. He says that the probabilities are very nearly the same from whatever table of observations they are computed; the accuracy of which remark may be best seen by a comparison with the examples above given.

The table here alluded to is the only one hitherto calculated for showing the probabilities of survivorship between any two lives. As that table, however, is applicable only to the valuation of annuities as deduced from the *Northampton* observations, and is, moreover, adapted to every decade only of human life, I shall point out an easy method whereby we may obtain a *near* value of such probability, which, though not strictly true, will be sufficiently correct if the two lives are between 10 and 70 years of age; at least

COROLLARY I.

§ 119. If the sum of any number of terms of that series be subtracted from the probability of the failing of the joint lives in that term, the difference will denote the probability that B will die before A in that time. Consequently (since it is certain that one or other of them will die before the end of m years), if the sum of the whole series be subtracted from unity, the difference, or $1-p$, will express the whole chance of B dying before A during the probable term of their joint continuance.

COROLLARY II.

§ 150. When the two lives are equal, the sum of the general series given in the problem is equal to $\frac{1}{2}$; that is, $p=\frac{1}{2}$.

SCHOLIUM.

§ 151. Since A and B may denote lives of any ages, and therefore the series given in the problem extends to all cases, whereby p becomes a general expression for the probability of survivorship between any two lives, yet, as different probabilities may hereafter arise in the same problem, and as much confusion may be created by the use of the same character to express different quantities, I here take the opportunity of observing that I shall denote the several probabilities of survivorships that may take place between any two of the three lives, A, B, C, by the following symbols, viz. :—

till we are possessed of more enlarged tables on this subject. Let the expectation of A's life, as found by § 52, be denoted by e; and let the expectation of B's life, as found in the same manner, be denoted by E; then will $p=\frac{E}{2e}$.

This formula is deduced from the assumption that the decrements of life are equal and uniform at every age, agreeably to the principles of M. de Moivre, which will be more fully explained in Chap. IX., and is one of the many instances of the utility and convenience of that celebrated hypothesis. The following examples, deduced from the *Northampton* observations, and compared with the values found by the Lemma, will show how far it may be relied on :—

Age of A.	Age of B.	Value by Hypothesis.	Value by Lemma.
20	30	·4228	·4231
20	40	·3452	·3429
20	50	·2692	·2650
20	60	·1976	·1931
30	40	·4082	·4094
30	50	·3182	·3170
30	60	·2336	·2302
40	50	·3897	·3938
40	60	·2862	·2882
50	60	·3671	·3756

A dying *before* $B=p$,
A ,, ,, $C=q$,
B ,, ,, $C=f$,

whence it follows, by Cor. 1, that the probability of

B dying *before* $A=1-p$,
C ,, ,, $A=1-q$,
C ,, ,, $B=1-f$.

LEMMA II.[1]

§ 152. To determine the probability that out of two given lives A and B, one of them in particular, A, shall die *after* the other.

SOLUTION.

It is manifest that this circumstance can take place in the first year only by the extinction of both the lives, A having died last; the probability of which $\frac{d_x}{l_x}\times\frac{d_y}{l_y}\times\frac{1}{2}$,[2] an expression which, for reasons that will hereafter appear, I shall make equal to $\frac{d_x}{l_x}-\frac{d_x d_y}{2l_x l_y}$.

But in the second and following years, the event may take place, (1.) by the extinction of both the lives in the year, A having died last; (2.) by the decease of A in the year, B having died in either of the preceding years. The probability of the first of these events happening in the second year is $\frac{d_{x1} d_{y1}}{2l_x l_y}$, and the probability of the second event happening in the same period is $\frac{d_{x1}}{l_x}(1-\frac{l_{y1}}{l_y})$: the sum of these two, therefore, or $\frac{d_{x1}}{l_x}-\frac{d_{x1} d_{y1}}{2l_x l_y}$, will denote the whole probability that A dies after B in the *second* year; and which expression, being added to the value just found for the first year, will give $\frac{d_x}{l_x}\times\frac{d_x d_y}{2l_x l_y}+\frac{d_{x1}}{l_x}-\frac{d_{x1} d_{y1}}{2l_x l_y}$, for the probability that A dies after B in *two* years.

In like manner, the probability of the first event happening in the third year is $\frac{d_{x2} d_{y2}}{2l_x l_y}$, and the probability of the second event happening in the same period is $\frac{d_{x2}}{l_x}\times(1-\frac{l_{y2}}{l_y})$; the sum of which, or $\frac{d_{x2}}{l_x}-\frac{d_{x2} d_{y2}}{2l_x l_y}$, will denote the whole probability of A dying after B in the *third* year: and which expression being added to the value just found for two years, will give $\frac{d_x}{l_x}-\frac{d_x\,d_y}{2l_x\,l_y}+\frac{d_{x1}}{l_x}-\frac{d_{x1}\,d_{y1}}{2l_x\,l_y}+\frac{d_x{}^2}{l_x}-\frac{d_{x2}\,d_{y2}}{2l_x\,l_y}$, for the probability that A dies after B in *three* years.

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[1] *Phil. Trans.* 1794, Part II. p. 224. Dodson, vol. iii. Ques. 19.

[2] See the note to § 145.

And so on with respect to the values for every subsequent year: and if we make m equal to the difference between the age of the oldest of the given lives, and that age in the Table of Observations *at which human life becomes extinct*,[1] we shall find that the series

$$\frac{d_x}{l_x}+\frac{d_{x1}}{l_x}+\frac{d_{x2}}{l_x}+\ldots\frac{d_{x+m-1}}{l_x}-\left[\frac{d_x\,d_y}{2l_x\,l_y}+\frac{d_{x1}\,d_{y1}}{2l_x\,l_y}+\frac{d_{x2}\,d_{y2}}{2l_x\,l_y}+\ldots\frac{d_{x+m-1}\,d_{y+m-1}}{2l_x\,l_y}\right]$$

will express the whole chance of A dying after B *during that period*; that is, during the probable term of their joint continuance.

But the first part of this series, or $\frac{d_x}{l_x}+\frac{d_{x1}}{l_x}+\frac{d_{x2}}{l_x}+\ldots\frac{d_{x+m-1}}{l_x}$, is evidently equal to $\frac{l_x-l_{x+m}}{l_x}=(1-\frac{l_{x+m}}{l_x})$; that is, equal to the probability that the life A shall fail in that period: and the latter part of the series, which is to be subtracted, is (by the preceding Lemma) equal to p: consequently the total required value for m years will be denoted by $(1-\frac{l_{x+m}}{l_x})-p$.

COROLLARY I.

§ 153. If the sum of any number of terms in the above series be subtracted from the probability of both the lives failing in that time, the difference will denote the probability that B will die after A during that period. Consequently, if the sum of the whole series above given be subtracted from $(1-\frac{l_{x+m}}{l_x})(1-\frac{l_{y+m}}{l_y})=1-\frac{l_{x+m}}{l_x}-\frac{l_{y+m}}{l_y}$,[1] (or the probability that both the lives fail in that period) the difference, or $(1-\frac{l_{x+m}}{l_x}-\frac{l_{y+m}}{l_y})-(1-\frac{l_{x+m}}{l_x}-p)=p-\frac{l_{y+m}}{l_y}$, will express the chance of B dying after A in that period.

---

[1] Mr. Morgan has taken this term equal only to the difference between the age of the oldest of the given lives and the age of *the oldest life* in the Table of Observations: whereby he has omitted the chance of one life dying after the other in the *last* year of their joint existence. The reader must bear this in mind when comparing the values deduced by this rule, with the values deduced from the rule given by Mr. Morgan. I would here observe, that the formula above given is not only the most correct of the two, but also renders the several expressions, in which it is afterwards used, much more simple than those which are deduced according to Mr. Morgan's assumption. And, agreeably to these principles, it will be found that the Table which Mr. Morgan has given, of the probabilities of one life *dying after* the other, is totally useless: since its application may always be avoided in practice; as will be evident from the method pursued in the investigation of the problems in this, and in the eighth chapter of the present work.

Should the reader be desirous of knowing the probability that A will die after B during the period of their joint continuance, *according to Mr. Morgan's assumption*, he will find that it is denoted by $(1-\frac{l_{x+m-1}}{l_x})-p$: that is, by substituting l_{x+m-1} for l_{x+m} in the formula which I have given in the text. In like manner, the probability of B dying after A in that period will, upon the same assumption, be denoted by $p-\frac{l_{y+m-1}}{l_y}$: that is, by substituting l_{y+m-1} for l_{y+m} in the formula given in the Scholium. And so of the other quantities there adduced.

COROLLARY II.

§ 154. When the two lives are both of the same age, p becomes equal to $\frac{1}{2}$; as already mentioned in the second corollary to the preceding Lemma: consequently the sum of the series in the present Lemma becomes also equal to $\frac{1}{2}$.

SCHOLIUM.

§ 155. In order to prevent the confusion of quantities alluded to in the Scholium given in § 151, I shall here denote the several probabilities of survivorship, which have been the subject of this Lemma, and which may arise between any two or three lives A, B, C, by the following symbols, viz.:

$$\text{A dying } \textit{after } \text{B}=1-p-\frac{l_{x+m}}{l_x}$$

$$\text{A} \;.\;.\;.\;.\; \text{C}=1-q-\frac{l_{x+m}}{l_x}$$

$$\text{B} \;.\;.\;.\;.\; \text{C}=1-f-\frac{l_{y+m}}{l_y}$$

whence it follows by cor. 1, that the probability of

$$\text{B dying } \textit{after } \text{A}=p-\frac{l_{y+m}}{l_y}$$

$$\text{C} \;.\;.\;.\;.\; \text{A}=q-\frac{l_{z+m}}{l_z}$$

$$\text{C} \;.\;.\;.\;.\; \text{B}=f-\frac{l_{z+m}}{l_z}$$

§ 156. It should here be remarked that the above expressions denote the respective probabilities of survivorship for m years only, or during the probable time of the joint continuance of the two lives; and that the values are deduced without any regard to seniority. Therefore, when A is the oldest of the two lives, the general expression in the Lemma will become $1-p$; because l_{x+m} becomes equal to 0, and consequently the fraction

---

[1] Since it is *certain* that one or other of the lives will be extinct at the end of m years, it follows that the quantity $\frac{l_{x+m}l_{y+m}}{l_x\,l_y}$, which arises from this product, will vanish altogether.

$\frac{l_{x+m}}{l_x}$ vanishes altogether. But, when A is the youngest of the two lives, that expression will not denote the *whole* probability of A dying after B, since there is a further chance of A dying, after having survived B. In order to determine this probability for the subsequent years, the series in the Lemma must be continued till the extinction of A's life: whence it will be found that the probability of A dying after B in $(m+1)$, $(m+2)$, $(m+3)$, &c., years will be respectively denoted by $1-p-\frac{l_{x1+m}}{l_x}$, $1-p-\frac{l_{x2+m}}{l_x}$, $1-p-\frac{l_{x3+m}}{l_x}$, &c.; to the utmost extent of A's life, at which period the expression becomes $1-p$.

In like manner, when B is the oldest of the two lives, the probability that B will die after A becomes equal to p: but, if B be younger than A, the general expression above given, $p-\frac{l_{y+m}}{l_y}$ denotes the probability of that event taking place during m years only, or during the probable time of their joint continuance. And the probability of the same event taking place in $(m+1)$, $(m+2)$, $(m+3)$, &c., years will be respectively denoted by $p-\frac{l_{y1+m}}{l_y}$, $p-\frac{l_{y2+m}}{l_y}$, $p-\frac{l_{y3+m}}{l_y}$, &c.; to the utmost extent of B's life, when the expression at length becomes equal to p. The same observations will apply to the other quantities above given.

PROBLEM XIX.[1]

§ 157. To find the value of a reversionary annuity on the life A, after the longest of two lives B and C, on condition that B dies *after* C.

SOLUTION.

The chance which A has of receiving this annuity at the end of any one year will depend on the continuance of his life to the end of such term, and on the extinction of both the lives B and C previous thereto; restrained however to the contingency that B dies last. It is this contingency which it is so difficult to represent in such a manner as to be generally useful. In the short space of one year, as I have before observed (§ 145), the error is not material by taking one *half* the product of the probabilities that the two lives shall fail in that period. "But, when the number of years and the difference between the ages of the two lives are considerable, those chances must vary in proportion; and therefore, unless the contingency is blended with another which shall very much diminish the probability of the event, the solution, by thus indiscriminately supposing the chances to be equal, must be rendered *extremely inaccurate*."[1]

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[1] Simpson's *Sup*. Prob. 34. Dodson, vol. iii. Ques. 30. Morgan, Prob. 27, cor. and in *Phil. Trans.* for 1794, page 240.

§ 158. If the probability of B dying after C in *one*, *two*, *three*, &c., years (as found by the second Lemma) be severally denoted by q_1, q_2, q_3, &c., the expectation which A has of enjoying the annuity at the end of those years will be represented, with a sufficient degree of accuracy,[2] $\frac{vl_{x1}q_1}{l_x}$, $\frac{v^2l_{x2}q_2}{l_x}$, $\frac{v^3l_{x3}q_3}{l_x}$, respectively. But, in this case, the true value of the reversionary annuity could not be expressed by less than m different series; and therefore would be wholly unfit for general use.

§ 159. It appears, from what has been said, that the chance of one life dying before or after another differs in *every* year of their joint existence; and that it is not capable of being represented by a *constant* quantity till the extinction of the oldest of such lives. *After* that period, however, the expectation which A has of enjoying the annuity at the end of any subsequent year may be determined sufficiently near for any useful purpose by the help of the preceding lemmata. Consequently, all that appears further necessary for the proper solution of the problem is such an expression as will approximate to the value of the chance that B will die after C during the several years of their *joint* continuance. Let such expression be denoted by g; (the value of which will be the subject of a future inquiry, see § 173): then will the value of the reversionary annuity, depending on the contingency mentioned in the problem, be determined in the following manner.

§ 160. It is manifest that the payment of the annuity in any one year depends on the continuance of the life A to the end of that year, and on the extinction of both the lives B and C previous thereto, B having died last; the probabilities of which events for the first, second, third, &c., years are respectively denoted by $\frac{l_{x1}}{l_x}(1-\frac{l_{y1}}{l_y})(1-\frac{l_{z1}}{l_z})g$, $\frac{l_{x2}}{l_x}(1-\frac{l_{y2}}{l_y})(1-\frac{l_{z2}}{l_z})g$, $\frac{l_{x3}}{l_x}(1-\frac{l_{z3}}{l_z})(1-\frac{l_{z3}}{l_z}g$, etc. Consequently, the sum of the expectations of receiving the annuity at the end of those periods respectively will, for the first m years,[1] be denoted by the following series:—

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[1] These are Mr. Morgan's own words when speaking of Mr. Simpson's method of solution: see *Phil. Trans.*, 1791, page 276. We shall find in the sequel, that he has fallen into the same error himself.

[2] Mr. Morgan says (*Phil. Trans.* for 1794, p. 238) that these quantities would give the *exact* value of the reversionary annuity; but he asserts this on the presumption that the values deduced from the Lemma give the *true* probabilities of survivorship for *every year* of human life. Whereas those values approximate only in proportion to the length of the series; and are incorrect in the first terms of such series, when there is any considerable inequality between the ages of the two lives. Nevertheless, if they could be at all

$$
\begin{aligned}
&vg\left(\frac{l_{x1}}{l_x}-\frac{l_{x1}\,l_{y1}}{l_x\,l_y}-\frac{l_{x1}\,l_{z1}}{l_x\,l_z}+\frac{l_x\,l_y\,l_{z|1}}{l_x\,l_y\,l_z}\right)+\\
&v^2g\left(\frac{l_{x2}}{l_x}-\frac{l_{x2}\,l_{y2}}{l_x\,l_y}-\frac{l_{x2}\,l_{z2}}{l_x\,l_z}+\frac{l_x\,l_y\,l_{z|2}}{l_x\,l_y\,l_z}\right)+\\
&v^3g\left(\frac{l_{x3}}{l_x}-\frac{l_{x3}\,l_{y2}}{l_x\,l_y}-\frac{l_{x3}\,l_{z3}}{l_x\,l_z}+\frac{l_x\,l_y\,l_{z|3}}{l_x\,l_y\,l_z}\right)+\\
&\quad\;\ldots\qquad\ldots\qquad\ldots\qquad\ldots\\
&v^mg\left(\frac{l_{x+m}}{l_x}-\frac{l_x\,l_{y|m}}{l_x\,l_y}-\frac{l_x\,l_{z|m}}{l_x\,l_z}+\frac{l_x\,l_y\,l_{z|m}}{l_x\,l_y\,l_z}\right).
\end{aligned}
$$

§ 161. *Case* 1. Let A be the oldest of the three lives. It is evident that in this case the series above given will denote the whole value of the reversionary annuity required; because in the mth year the life A becomes extinct, and all the subsequent terms of the series vanish. But the above series is equal to $a_x-a_{xy}-a_{xz}+a_{xyz}$; that is, equal to the value of a reversionary annuity on the life A after the longest of two lives B and C, multiplied by the chance that B dies after C.

According to the method of solution adopted by Mr. Simpson and Mr. Morgan, the values (in all those cases where A is the oldest life) will be precisely the same, whatever be the difference of age between B and C, whether the contingency depends on B dying after C, or C dying after B. Thus, the value of an annuity on a life aged 78 to be entered upon at the extinction of two lives aged 15 and 75 (which is one of the cases given by Mr. Morgan) is the same whether the contingency depends on the younger life dying after the elder, or on the elder dying after the younger. But it must be evident that an annuity depending on the former contingency is worth more than a similar annuity depending on the latter contingency. For the time of A's coming into possession is the same in both cases, viz., on the extinction of *both* the lives; therefore the value of the annuity will be affected only by the contingency of one life dying after the other.

Mr. Morgan, in attempting to give a more correct solution for the value of such annuities after the extinction of the oldest life, has overlooked the most material part of the process, which is to obtain a more accurate expression for the value of such annuities *during the joint continuance* of all the lives. I am aware that Mr. Morgan asserts that, by taking it as an *equal* chance whether B dies before or after C in any given period, the

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rendered fit for practical purposes, they would enable us to approximate more nearly to the true value of the reversionary annuity than the inaccurate method hitherto adopted.

[1] I would here observe, once for all, that in this and the two subsequent problems I take m to denote the number of years between the age of the oldest life involved in the question and that age in the table of observations when human life becomes extinct; consequently, the value of m will vary according to the three cases given in these problems.

value of the annuity which results from this assumption will be sufficiently near the *true* value for any useful purpose; and he has given some examples with a view to prove the accuracy of his remark. I shall here, however, take the opportunity of observing, that what he calls the *true* value is only an *approximation*, which differs most from the true value in those very cases where it is most wanted as a test. He has deduced certain values from a false hypothesis; and afterwards, assuming these values as if correct, endeavours to prove that another method of approximation used by him is accurate because it agrees with these assumed values. Now, from what has been said in the preceding pages, I think it must be evident that, when there is any considerable difference between the ages of the two lives, the value of the probability deduced by the lemmata will not be the correct value for every year of human life; neither will the method of proceeding, alluded to in § 158, enable us in such cases to obtain the true value of the reversionary annuity.

These observations apply to the table inserted by Mr. Morgan in the *Phil. Trans.* for 1794, p. 234, and to the examples given by him in page 239 of the same volume. Mr. Morgan says, "that the approximations and exact values do not differ much from each other till the last years of the oldest life:" but the fact is that they differ nearly in the same ratio through the whole time of the joint continuance of the two lives. His own remarks prove the inaccuracy of his method of reasoning.

§ 162. *Case* 2. Let B be the oldest of the three lives. In this case, the above series will denote the value of the reversionary annuity for the first *m* years (*m now* denoting the number of years between the age of B and that age in the table of observations when human life becomes extinct); and the sum of it will be found equal to $g(a_{x)m}-a_{xy}-a_{xz)m}+a_{xyz})$. For, the first and third of these perpendicular series being continued to *m* terms only, the sum of those terms will, by Prob. I. cor. 4, be accurately represented by the characters here given; and the second and fourth perpendicular series evidently denote the whole value of the annuities on those lives respectively; since in the *m*th year the life B becomes extinct, and all the subsequent terms of those series vanish.

Now, in order to determine the expectation of receiving the several rents in the remaining years of A's life, it should be observed that the payment of the annuity in any one year depends on the continuance of the life A to the end of that year, and on the probability that B dies after C previous thereto: but with the probability, that B will die after C in the $(m+1)^{\text{st}}$ and all the subsequent years, is (by the scholium to the second Lemma) denoted by the constant quantity $(1-f)$.[1] Consequently the sum of the expectations of receiving the $(m+1)^{\text{st}}$, $(m+2)^{\text{nd}}$, $(m+3)^{\text{rd}}$, &c., year's rents will be expressed by the series $\frac{v^{m+1}(1-f)\,l_{x1+m}}{l_x}+\frac{v^{m+2}(1-f)\,l_{x2+m}}{l_x}+\frac{v^{m+3}(1-f)\,l_{x3+m}}{l_x}+$ &c.; which, being continued for all the subsequent years of A's life, will show the true value of all the rents to be received after m years. But the sum of this series is, by Prob. I., cor. 3, equal to $(1-f)a_{x(m}$; and which, being added to the first m terms of the several collateral series above found, will make the total value in this case equal to $g(a_{x)m}-a_{xy}-a_{xz)m}+a_{xyz})+(1-f)a_{x(m}$. But since $a_{x)m}=a_x-a_{x(m}$, and $a_{xz)m}=a_{xz}-a_{xz(m}$, as appears from Prob. I. cor. 4, it follows that this value may be more conveniently expressed by $g(a_x-a_{xy}-a_{xz}+a_{xyz})+(1-f-g)\times a_{x(m}+g.a_{xz(m}$.

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[1] Because in the general expression there given, l_{y+m} becomes equal to 0 when B is the oldest life; and the fraction $\frac{l_{y+m}}{l_y}$ consequently vanishes.

I would here observe, that the rule given by Mr. Simpson for the solution of this case is expressed by $(a_x-2a_{xy}+a_{xyy})\frac{E}{n2}$, where E denotes the expectation of the life B, as deduced from the rule in § 52, and where n denotes the expectation of the life C, as deduced from the same rule.

When C is the oldest life, this value must be deducted from the whole value of an annuity on the life A after the longest of the two lives, B and C, agreeably to the method which will be explained in the note to § 174.

§ 163. *Case* 3. Let C be the oldest of the three lives. The value of all the rents for the first m years (m *now* denoting the number of years between the age of C and that age in the table of observations when human life becomes extinct) may in this case, as in the last, be denoted by the first m terms of the series given in the problem, and the sum of which will now be equal to $g[a_{x)m}-a_{xy)m}-a_{xz}+a_{xyz}]$. For the first and second of those perpendicular series being continued to m terms only, the sum of those terms will be truly denoted by the characters here given; but the last two perpendicular series evidently denote the value of annuities on the *whole continuance* of those lives respectively, because in the mth year the life C is extinct.

Now, in order to determine the expectation of receiving the several rents in the remaining years of A's life, it should be observed that the probability of B dying after C in $(m+1)$, $(m+2)$, $(m+3)$, &c., years is, by the scholium to the second Lemma, respectively denoted by $(1-f-\frac{l_{y1+m}}{l_y})$, $(1-f-\frac{l_{y2+m}}{l_y})$, $(1-f-\frac{l_{y3+m}}{l_y})$, &c. Consequently, since the payment of the annuity in any one year depends on the continuance of A's life to the end of that year, and on B having died after C previous thereto, it follows that if these values be severally multiplied by $\frac{l_{x1+m}}{l_x}$, $\frac{l_{x2+m}}{l_x}$, $\frac{l_{x3+m}}{l_x}$, ... (or the probability that A will live $(m+1)$, $(m+2)$, $(m+3)$, &c., years respectively) the products thence arising will express the whole chance of receiving each year's rent respectively. And these products, being again multiplied by the present value of £1 due at the end of those respective periods, will give the expectations of receiving such sum at the end of those years; and which will be expressed by the following series:—

$$
\begin{array}{l}
v^{m+1}\left[(1-f)\frac{l_{x1+m}}{l_x}-\frac{l_x l_{y|1}}{l_x l_y}\right]+ \\
v^{m+2}\left[(1-f)\frac{l_{x2+m}}{l_x}-\frac{l_x l_{y|2}}{l_x l_y}\right]+ \\
v^{m+3}\left[(1-f)\frac{l_{x3+m}}{l_x}-\frac{l_x l_{y|3}}{l_x l_y}\right]+ \\
\quad\cdots\qquad\cdots\qquad\cdots\qquad\cdots
\end{array}
$$

But the sum of these two perpendicular series, continued to the utmost extent of A's life, is, by Prob. I. cor. 3, equal to $(1-f)a_{(m}-a_{xy(m}$, and which, being added to the first m terms of the several collateral series above found, will make the total value of the annuity in this case equal to $g(a_{x)m}-a_{xy)m}-a_{xz}+a_{xyz})+(1-f)a_{x(m}-a_{xy(m}$, an expression which, from what has been said in the last case, may be more conveniently denoted by $g(a_x-a_{xy}-a_{xz}+a_{xyz})+(1-f-g)a_{x(m}-(1-g)a_{xy(m}$.

COROLLARY.

§ 164. If the two lives in possession are both of the same age, B, then f and g each become equal to $\frac{1}{2}$, and the quantities $a_{xy(m}$ and $a_{xz(m}$ vanish altogether; consequently the expression in each becomes $\frac{1}{2}(a_x-2a_{xy}+a_{xyy})$; that is, equal to half the value of a reversionary annuity on the life A after the longest of two equal lives, B and B.

But if the two lives in possession are of the same age with the life A in reversion, this expression will become $\frac{1}{2}(a_x-2a_{xx}+a_{xxx})$.

PROBLEM XX.[1]

§ 165. To find the value of a reversionary annuity on the life A, after two joint lives, BC, on condition that B dies *before* C.

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[1] Simpson's *Sup.*, Prob. 35. Dodson, vol. iii. Ques. 49. Morgan, Prob. 27; and in *Phil. Trans.* for 1794, p. 235.

SOLUTION.

The payment of this annuity at the end of any one year depends on the continuance of the life A to the end of that year, and on the extinction of the life B before C previous thereto. As this latter contingency cannot be conveniently represented with accuracy for every year of human life, it becomes necessary, as in the last problem, to have recourse to an approximation during the number of years that the lives have a chance of continuing together. Therefore let the same symbols be used as in that problem; and let us consider the payment of the annuity, during the first m years, as depending in each year on one or other of two events; 1. that A lives to the end of the year, and that B and C both fail previous thereto, B having died first; 2. that only B dies previous thereto, and that A and C both live. The probability of the happening of the first event in the first year is $\frac{l_{x1}}{l_x}(1-\frac{l_{y1}}{l_y})(1-\frac{l_{z1}}{l_z})(1-g)$;[1] and the probability of the happening of the second event in the same period is $(1-\frac{l_{y1}}{l_y})\frac{l_{x1}\,l_{z1}}{l_x\,l_z}$; therefore the sum of these two, reduced to simple terms and multiplied by v, will give $v\left[\frac{l_{x1}}{l_x}-\frac{l_x l_y {}_{\|1}}{l_x l_y}-\left(\frac{l_{x1}}{l_x}-\frac{l_x l_y {}_{\|1}}{l_x l_y}-\frac{l_x l_z {}_{\|1}}{l_x l_z}+\frac{l_x l_y l_z {}_{\|1}}{l_x l_y l_z}\right)g\right]$ for the total value of the expectation of receiving the first year's rent. By a similar method of reasoning we may find the expectation of receiving the second, third, &c., year's rents; and the sum of all these expectations for the first m years, will be denoted by the following series:—

$$
\begin{array}{l}
v\left[\frac{l_{x1}}{l_x}-\frac{l_x l_y {}_{\|1}}{l_x l_y}-\left(\frac{l_{x1}}{l_x}-\frac{l_x l_y {}_{\|1}}{l_x l_y}-\frac{l_x l_z {}_{\|1}}{l_x l_z}+\frac{l_x l_y l_z {}_{\|1}}{l_x l_y l_z}\right)g\right]+\\
v^2\left[\frac{l_{x2}}{l_x}-\frac{l_x l_y {}_{\|2}}{l_x l_y}-\left(\frac{l_{x2}}{l_x}-\frac{l_x l_y {}_{\|2}}{l_x l_y}-\frac{l_x l_z {}_{\|2}}{l_x l_z}+\frac{l_x l_y l_z {}_{\|2}}{l_x l_y l_z}\right)g\right]+\\
v^3\left[\frac{l_{x3}}{l_x}-\frac{l_x l_y {}_{\|3}}{l_x l_y}-\left(\frac{l_{x3}}{l_x}-\frac{l_x l_y {}_{\|3}}{l_x l_y}-\frac{l_x l_z {}_{\|3}}{l_x l_z}+\frac{l_x l_y l_z {}_{\|3}}{l_x l_y l_z}\right)g\right]+\\
\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
v^m\left[\frac{l_{x+m}}{l_x}-\frac{l_x l_y {}_{\|m}}{l_x l_y}-\left(\frac{l_{x+m}}{l_x}-\frac{l_x l_y {}_{\|m}}{l_x l_y}-\frac{l_x l_z {}_{\|m}}{l_x l_z}+\frac{l_x l_y l_z {}_{\|m}}{l_x l_y l_z}\right)g\right]
\end{array}
$$

§ 166. *Case* 1. Let A be the oldest of the three lives. It is evident that in this case the series above given will express the whole value of the reversionary annuity required: because in the mth year the life A becomes extinct, and all the subsequent terms of the series vanish. But the above series is in such case evidently equal to $a_x-a_{xy}-g(a_x-a_{xy}-a_{xz}+a_{xyz})$.

---

[1] Since g denotes the chance that B will die *after* C in every year of their joint existence, it follows that $1-g$ will denote the chance that B will die *before* C in the same periods.

§ 167. *Case* 2. Let B be the oldest of the three lives. In this case, the above series will denote the value of the reversionary annuity for the first m years, and the sum of it will be found to be $a_{x)m}-a_{xy}-g(a_{x)m}-a_{xy}-a_{xz)m}+a_{xyz})$. In the subsequent years, the chance of receiving the annuity will depend on the continuance of the life A, and on the probability of B dying before C: therefore the value of the annuity for the remaining years will be denoted by the sum of the series $\frac{fl_{x_1+m}}{l_x}v^{m+1}+\frac{fl_{x_2+m}}{l_x}v^{m+2}+\frac{fl_{x_3+m}}{l_x}v^{m+3}\ldots=fa_{x(m}$. Consequently the total value of the required annuity will be $a_x-a_{xy}-g(a_x-a_{xy}-a_{xz}+a_{xyz})-(1-f-g)\times a_{x(m}-ga_{xz(m}$.

§ 168. *Case* 3. Let C be the oldest of the three lives. In this case, the above series will also denote the value of the reversionary annuity for the first m years; the sum of which will now become equal to $a_{x)m}-a_{xy)m}-g(a_{x)m}-a_{xy)m}-a_{xz}+a_{xyz})$: and the value of the annuity for the remaining years will be truly expressed, as in the last case, by $fa_{x(m}$. Consequently the total value of the annuity required will be $a_x-a_{xy}-g\,(a_x-a_{xy}-a_{xz}+a_{xyz})-(1-f-g)a_{x(m}+(1-g)a_{xy(m}$.

COROLLARY.

§ 169. If the two lives in possession are both of the same age B, then f and g will each (as in the corollary to the last problem) become equal to $\frac{1}{2}$, and the quantities $a_{xy(m}$ and $a_{xz(m}$ vanish altogether: consequently the value of the reversionary annuity in each case is expressed by $\frac{1}{2}(a_x-a_{xyy})$.

But when the two lives in possession are of the same age as the life A in reversion, this expression will become equal to $\frac{1}{2}(a_x-a_{xxx})$.

SCHOLIUM.

§ 170. If the value found according to either of the three cases in this problem be added to the value found according to the corresponding case in the preceding problem, their sum will be equal to a_x-a_{xy}. Consequently, if the value of the annuity in either case of the preceding problem be once determined, the value of the same annuity, according to the corresponding case in the present problem, may be easily found (provided the ages of the three lives A, B, C, are the same in both instances) by subtracting the former value from a_x-a_{xy}: and *vice versa*.

This is almost self-evident: for, taking the contingencies of both problems, it is manifest that the annuity is certain to be enjoyed by A after the decease of B, provided he lives so long. Consequently the sum of the values found by both problems must, in each case, be equal to the value of a reversionary annuity on the life A after the decease of B. The truth of this Scholium is indeed evident on inspection.

PROBLEM XXI.[1]

§ 171. B and C possess an annuity between them; which, if B survives C, is afterwards to be equally divided between A and B during their joint existence, and then is to go entirely to the survivor for his life: To find the value of the share of A in this annuity.

SOLUTION.

Since A is not to receive anything if C survives B, it is evident that he is entitled to half the value of a reversionary annuity on the joint lives AB after the extinction of the life C; and also to the whole value of a reversionary annuity on his own life after the extinction of the life B, provided B dies after C. The former value is, by the Scholium in § 76, equal to $\frac{1}{2}(a_{xy}-a_{xyz})$; and the latter value may be found by Problem XIX., according as A, B, or C, is the oldest of the three lives: whence the following values may be readily deduced, for the three cases there mentioned.

That is, when A is the oldest of the three lives, the required value will be equal to $g(a_x-a_{xz})+(\frac{1}{2}-g)(a_{xy}-a_{xyz})$.[2] When B is the oldest of the three lives, it will be equal to $g(a_x-a_{xz})+(\frac{1}{2}-g)(a_{xy}-a_{xyz})+(1-f-g)a_{x(m}+ga_{xz(m}$. When C is the oldest of the three lives, it will be equal to $g(a_x-a_{xz})(\frac{1}{2}-g)(a_{xy}-a_{xyz})+(1-f-g)a_{x(m}-(1-g)a_{xy(m}$. When the two lives in possession are of the same age B, it will be equal to $\frac{1}{2}(a_x-a_{xy})$.[3] And when all the lives are of the same age A, it will be equal to $\frac{1}{2}(a_x-a_{xx})$.[4]

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[1] Simpson, Prob. 29, and *Sup.* Prob. 36.

[2] If we take g to denote $\frac{1}{2}$ (that is, if we suppose with Mr. Morgan indiscriminately that it is an even chance whether B dies before or after C in any period of their joint existence, whatever be the difference of age between the two lives) then will this formula become equal to $\frac{1}{2}(a_x-a_{xz})$: and this is the method of solution adopted also by Mr. Simpson. Mr. Morgan, however, in his hurry to attack M. De Moivre's hypothesis, has inadvertently called this formula an *absurd* one; and says that the error arises from Mr. Simpson's having been misled by that *hypothesis* in determining the probability of one life dying after the other: see *Phil. Trans.* vol. lxxxi. p. 276. The present investigation, *from the real probabilities of life,* will show that accusation to be unfounded. Mr. Morgan has wholly misstated the case, as he will readily perceive on re-perusing what he has there written: he has confounded Mr. Simpson's 34th and 36th Problems together; and thereby brought an unmerited censure upon that author. The error does not arise from the use of De Moivre's hypothesis (as he would *wish* us to believe), but from the inaccurate method which he, *in common with Mr. Morgan,* has adopted in order to express the chance of one life dying before or after the other during the probable time of their joint continuance; and the same *absurd* formula (if I may retort the self-confuting charge) will equally arise, as I have above observed, even according to Mr. Morgan's own method of solution.

[3] This formula will not differ much from the true value when the two lives in possession are *nearly* of the same age.

[4] Many other cases of survivorships might be produced which involve the contingencies

GENERAL SCHOLIUM.

§ 172. It now remains only to determine the value of g in order to obtain the proper solution of these three problems; and if the chance of one life dying before or after another in every year of their joint existence, were it in a constant ratio, we should find no difficulty therein. But since (in computing from the real probabilities of life) this chance is continually varying, we must have recourse to an approximation towards the mean value of such ratio.

§ 173. Now I have found, from a number of repeated trials, that the value of g may, when B is the youngest of the two lives B and C, be safely expressed by $\frac{(1-f)l_y-l_{y+m}}{l_y-l_{y+m}}$, and that when C is the youngest of the two lives B and C it may be safely expressed by $\frac{(1-f)l_z}{l_z-l_{z+m}}$: which, though not in all cases strictly correct,[1] will come nearer to express the true value of the reversionary annuity than by making g indiscriminately equal to $\frac{1}{2}$, whatever be the ages of B and C: and may be used till its true value be more correctly determined. But should a more accurate expression for the value of g be hereafter found, the general solution of the three problems will not be at all affected thereby, since we may give to g all possible values. I shall now insert a few examples, in order to show the use and application of these several formulæ.

§ 174. *Example* 1. What is the value of an annuity on the life of A aged 60 after B aged 40, provided B dies *after* another life C aged 20: interest being reckoned at 4 per cent., and the probabilities of living as at *Northampton?*

Here we shall have $a_x=9{\cdot}039$, $a_{xy}=7{\cdot}490$, $a_{xz}=7{\cdot}995$, $a_{xyz}=6{\cdot}722$, and $g={\cdot}395$: consequently the value of the reversionary annuity required will in this case be equal to ${\cdot}276\times{\cdot}395={\cdot}109$. But if B had been 20 and C 40 years of age, we should have $g={\cdot}605$: consequently the value of the reversionary annuity would in this case be equal to ${\cdot}276\times{\cdot}605={\cdot}167$.

In like manner, if A had been 20, B 60, and C 40 years of age, we should have $a_x=16{\cdot}033$, $a_{xy}=7{\cdot}995$, $a_{xz}=10{\cdot}924$, $a_{xyz}=6{\cdot}722$, $m=37$, $g={\cdot}354$, and $f={\cdot}7118$: consequently the value of the reversionary

mentioned in Problems XIX. and XX., and which are solved by the help of those problems in the manner here stated. But, after this investigation, I presume the reader will not find any difficulty in the solution of any other question of this kind that may occur in practice.

[1] It cannot be strictly correct, because the *true* value of f cannot be deduced by the method pursued in the lemmata, as I have already observed in the note in p. 74.

annuity would in this case be equal to 3·836 × ·354 − ·067 + ·026 = 1·317. But if B had been 40 and C 60 years of age, the value would in such case come out equal to 2·519.

It may here be useful to remark, that if the value of the annuity on the life A after the longest of the two lives B and C (provided B dies after C) be once found, we may readily determine the value of the same annuity, on the contingency that C dies after B, by subtracting the value thus found from the whole value of the reversionary annuity on the life A after the longest of the two lives B and C. Thus, the value found by the example given in the text being equal to 1·317, and the latter value here alluded to being equal to 3·836, it follows that 3·836 − 1·317 = 2·519 will be the value required.

In order that the reader may see the difference in the results, according to Mr. Morgan's formulæ and those which are here given, I shall insert the following comparative values of annuities on the life A after B, provided B dies after C, deduced from Northampton tables, and reckoning interest at 4 per cent.: which difference arises from the inaccurate method, adopted by Mr. Morgan, of taking it as an *equal* chance in all cases that B will die after C, whatever be their difference of age.

Age of A.	Age of B.	Age of C.	Value by Baily.	Value by Morgan.
60	40	20	·109	·138
60	20	40	·167	·138
20	60	40	1·317	1·717
20	40	60	2·519	2·120
40	60	20	·519	·716
40	20	60	·986	·789

§ 175. *Example* 2. What is the value of an annuity on the life of A aged 60 after B aged 40, provided B dies *before* another life C aged 20: interest at 4 per cent., and the probabilities of living as at *Northampton?*

Here we shall have $a_x - a_{xy} = 9·039 - 7·490 = 1·549$: consequently, 1·549 − ·109 = 1·440 will (agreeably to what has been said in the Scholium in § 170) be the value of the reversionary annuity in this case required. But if A had been 20 and C 60 years of age, the value of the reversionary annuity would, in such case, be equal to 5·109 − 2·519 = 2·590.

CHAPTER VI.

ON ASSURANCES.

§ 176. In the preceding chapters I have considered the present value of sums of money as depending on the existence of any given lives, or on any particular survivorship between them; and, in the solution of the different problems relative thereto, have had regard only to the probability of the *living* of those persons on whom the annuity was considered as depending. I come now, however, to treat of those cases where it is required to find the value of annuities, or of sums of money, depending on the *extinction* of any lives; or, in other words, to treat of the value of *Assurances on lives* :[1] a term applied to that compact whereby security is granted for the payment of an annuity or sum of money on the expiration of the lives on which the grant is made, in consideration of such a previous payment made to the assurer, as is accounted a sufficient compensation for the chance of loss to which he exposes himself. The value of this payment (commonly called the *Premium*), in all the principal cases which arise out of this subject, it is my object in the present chapter to determine.

§ 177. It may here, however, be necessary previously to observe that the method to be pursued, for determining the value of any *sum* depending on the extinction of any given lives, will be materially different from that which is pursued for determining the value of any *annuity* under the same circumstances. In the latter case, the expectant is to receive *several* yearly rents, the expectation of receiving *each* of which is independent of his expectation of receiving any other of them. But in reversionary *sums* the case is very different: for here, only *one gross sum* is to be received at the extinction of the given lives; and therefore the expectation of receiving it at the end of any one year will depend on its *not* having been received in any preceding year : or, which is the same thing, the chance of receiving the sum at the end of any year will be compounded of the probability of the given lives failing *in* that year, and of their having continued through all the preceding years. This, however, will more fully appear in the following investigations.

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[1] The term *Assurance* is usually applied only to the value of annuities or sums of money to be paid after the *extinction* of any given lives; but it may, with equal propriety, be applied to the value of those annuities which are paid *during the existence* of any given lives; and which have been the subject of the preceding chapters. For, if I give a sum of money for the grant of an annuity during the continuance of any given lives, I give such sum in order to have the annuity *assured* to me; which, without this warranty, would be precarious and uncertain. As I am ignorant of any other word, but such as would be equally ambiguous and indefinite, I have used the term *assurance*, in its most common acceptation, to express the values treated of in the present chapter.

PROBLEM XXII.[1]

§ 178. To determine the present value of a given sum payable at the end of the year in which any number of lives become extinct.

SOLUTION.

Let us in the present investigation confine the case to three joint lives A B C, whose probabilities of continuing 1, 2, 3, &c., years are as expressed in § 24: and let the given sum be denoted by s. Now, the present value of such sum, certain to be received at the end of one year, is equal to sv: but as the chance of receiving this sum at the end of the first year depends on the joint lives failing in that year, the probability of which is $\frac{l_x l_y l_z - l_x l_y l_{z|1}}{l_x l_y l_z}$, we must multiply the present value of the sum above mentioned by this probability; which will give $sv\left(\frac{l_x l_y l_z - l_x l_y l_{z|1}}{l_x l_y l_z}\right)$ for the true value of the first year's expectation, or of the chance of receiving such sum at the end of the first year.

In like manner, the present value of the given sum, certain to be received at the end of two years, is equal to sv^2: but as the chance of receiving such sum at the end of the second year depends on the joint lives failing *in* that year, the probability of which (by § 27) is $\frac{l_x l_y l_{z|1} - l_x l_y l_{z|2}}{l_x l_y l_z}$ we must multiply the present value of the sum above mentioned by the probability; which will give $sv^2\left(\frac{l_x l_y l_{z|1} - l_x l_y l_{z|2}}{l_x l_y l_z}\right)$ for the true value of the expectation of receiving such sum at the end of the second year.

And by a similar method of proceeding it will be found that, since the chance of receiving the given sum at the end of the third year depends on the joint lives failing *in* that year, the probability of which is $\frac{l_x l_y l_{z|2} - l_x l_y l_{z|3}}{l_x l_y l_z}$, we shall have $sv^3\left(\frac{l_x l_y l_{z|2} - l_x l_y l_{z|3}}{l_x l_y l_z}\right)$ for the true value of the expectation of receiving such sum at the end of the third year. And so on for every subsequent year of human life; the sum of all which values will be the required present value of the given sum.

§ 179. But the sum of these quantities, reduced to their simplest terms, is equal to the two following series: $\frac{sv}{l_x\,l_y\,l_z}(l_x\,l_y\,l_z+vl_x\,l_y l_{z|1}+v^2 l_x l_y l_{z|2}+$ $v^3 l_x\,l_y\,l_{z|3}\ldots)-\frac{s}{l_x\,l_y\,l_z}(vl_x\,l_y\,l_{z|1}+v^2 l_x\,l_y\,l_{z|2}+v^3 l_x\,l_y\,l_{z|3}\ldots)$; the former of which is equal to $sv(1+a_{xyz})$; and the latter, which is to be subtracted, is equal to sa_{xyz}. Consequently the total present value required becomes $s[v(1+a_{xyz})-a_{xyz}]=s[v+(v-1)a_{xyz}]$;[1] and though this case is confined to three lives, yet it is easy to see that the method of solution is general, and will apply to any other number of lives; whence the following rule:—

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[1] Simpson, Prob. 21, and *Sup.* Prob. 26. De Moivre, Prob. 16, Dodson, vol. ii. Ques. 89. Morgan, Prob. 8. Price, Note (E). On referring, however, to the first three authors here alluded to, it will be found that their investigations are erroneous; inasmuch as they consider a given *sum* as equivalent to the perpetuity of an *annuity* (commencing immediately) equal to the interest of such sum. See the Scholium in § 200.

§ 180. *Multiply the value of an annuity on the given lives by the rate of interest, and subtract the product from unity; divide the remainder by the amount of £1 in one year, and the quotient multiplied by the given sum will be the value required.*

For examples of the use and application of this problem, see Question 27 in Chapter XII.

COROLLARY I.

§ 181. In this problem I have considered the present value of the reversionary sum as depending on the extinction of the given lives in *whatever* year that may happen during the probability of their joint continuance; but if this contingency is *Deferred* for any number of years $(=m)$, that is, if we wish to ascertain the present value of a given sum payable on the failure of such lives, provided that shall happen *after* the given period, the formula will be materially altered. For, by pursuing the same steps as in the problem, it will be found that the expectation of receiving the given sum, at the end of the $(m+1)^{\text{st}}$, $(m+2)^{\text{nd}}$, $(m+3)^{\text{rd}}$, &c., years, to the utmost extent of human life, will be denoted by the series

$$s\left[\frac{l_x\,l_y\,l_{z|m}-l_x\,l_y\,l_{z|m1}}{l_x\,l_y\,l_z}v^{m+1}+\frac{l_x\,l_y\,l_{z|m1}-l_x\,l_y\,l_{z|m2}}{l_x\,l_y\,l_z}v^{m+2}+\right.$$
$$\left.\frac{l_x\,l_y\,l_{z|m2}-l_x\,l_y\,l_{z|m3}}{l_x\,l_y\,l_z}v^{m+3}\ldots\right],$$

which may be more conveniently divided into the two following series:—

$$\frac{sv^m}{l_x\,l_y\,l_z}(vl_x\,l_y\,l_{z|m1}+v^2 l_x\,l_y\,l_{z|m2}+v^3 l_x\,l_y\,l_{z|m3}+\ldots)-$$
$$\frac{sv^m}{l_x\,l_y\,l_z}(vl_x\,l_y\,l_{z|m1}+v^2 l_x\,l_y\,l_{z|m2}+v^3 l_x\,l_y\,l_{z|m3}+\ldots).$$

The latter of these is, by Prob. I., cor. 3, equal to $-sa_{xyz(m}$, and the former is equal to $sv\left[\frac{l_x\,l_y\,l_{z|m}}{l_x\,l_y\,l_z}v^m+a_{xyz(m}\right]$. consequently the required present value of the deferred assurance will be equal to $sv\left[\frac{v^{m+1}l_x\,l_y\,l_{z|m}}{l_x\,l_y\,l_z}+(v-1)a_{xyz(m}\right]$;[1] and whence the following rule:[2]—

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[1] Since $v=\frac{1}{1+r}$ and $v-1=-r$, it follows that $v+(v-1)a_{xyz}=\frac{1-ra_{xyz}}{1+r}$, according to the subsequent rule laid down by the author.—EDITOR.

§ 182. *Multiply the value of a Deferred annuity on the given lives by the rate of interest; subtract the product from the expectation that the given lives shall receive £1 at the end of the given term; and divide the difference by the amount of £1 in a year: the quotient thence arising, being multiplied by the given sum, will produce the value required.*

For examples of the use and application of this corollary, see Question 28 in XII.

COROLLARY II.

§ 183. But if the contingency is *Temporary*, that is, if we wish to ascertain the value of a given sum payable on the failure of such lives, provided that shall happen *within a given term* ($=m$), it will be found by taking the first m terms of the series given in the problem. Whence, such present value will in this case be denoted by the series $s\left[\frac{v(l_x\,l_y\,l_z-l_x\,l_y\,l_{z\,|1})}{l_x\,l_y\,l_z}+\frac{v^2(l_x\,l_y\,l_{z\,|1}-l_x\,l_y\,l_{z\,|2})}{l_x\,l_y\,l_z}+\frac{v^3(l_x\,l_y\,l_{z\,|2}-l_x\,l_y\,l_{z\,|3})}{l_x\,l_y\,l_z}+\ldots\frac{v^m(l_x\,l_y\,l_{z\,|m-1}-l_x\,l_y\,l_{z\,|m})}{l_x\,l_y\,l_z}\right]$: which may be more conveniently divided into the two following ones: $\frac{sv}{l_x\,l_y\,l_z}\left[l_x\,l_y\,l_z+vl_x\,l_y\,l_{z\,|1}+v^2l_x\,l_y\,l_{z\,|2}+v^3l_x\,l_y\,l_{z\,|3}+\ldots v^{m-1}l_x\,l_y\,l_{z\,|m-1}\right]-\frac{s}{l_x\,l_y\,l_y}[vl_x\,l_y\,l_{z\,|1}+v^2l_x\,l_y\,l_{z\,|2}+v^3l_x\,l_y\,l_{z\,|3}+\ldots v^ml_x\,l_y\,l_{z\,|m}]$. But the first of these is equal to $sv(1+a_{xyz)m-1})$;[3] and the second, which is to be subtracted, is evidently equal to $sa_{xyz)m}$; consequently the required value of the assurance *for the given term* might be denoted by the formula $s[v(1+a_{xyz)m-1})-a_{xyz)m}]$.

But, since $a_{xyz)m-1}+\frac{v^ml_x\,l_y\,l_{z\,|m}}{l_x\,l_y\,l_z}$ is equal to $a_{xyz)m}$[4] whereby $a_{xyz)m-1}$ becomes equal to $a_{xyz)m}-\frac{v^ml_x\,l_y\,l_{z\,|m}}{l_x\,l_y\,l_z}$; and since, by Prob. I. cor. 4,

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[1] See Editor's note to § 179; the same applies to this case.—EDITOR.

[2] Since $a_{xyz(m}$ is, by Prob. I. cor. 3, equal to $a_{xyz]m}\,\frac{v^m\,l_x\,l_y\,l_{z(|m}}{l_x\,l_y\,l_z}$; it is obvious that the present value, in the case of *single or joint* lives, might be more conveniently expressed by $sv^m[v+(v-1)a_{xyz(|m}]\frac{l_x\,l_y\,l_{z|m}}{l_x\,l_y\,l_z}$: and it is from this formula that I have deduced the rule in Question 28, Chapter XII. But that rule will not extend to *all* cases.

[3] The new character $a_{xyz)m-1}$ denotes the value of a temporary annuity on the joint lives ABC for $m-1$ years only, and is introduced here merely to show the steps of the process.

[4] As is evident by continuing the series which this quantity expresses.

$a_{xyz)m}$ is equal to $a_{xyz}-a_{xyz(m}$; we may render the above formula more convenient for general use by substituting such value therein instead of $a_{xyz)m}$: whereby it will become equal to $sv[1-ra_{xyz}-(\frac{l_x\,l_y\,l_{z|m}\,v^m}{l_x\,l_y\,l_z} - ra_{xyz(m})]$: and whence the following rule :—

§ 184. *From the present value of the assurance of the given sum payable on the extinction of the given lives, subtract the present value of the same assurance Deferred for the given term: the difference will be the value required.*

Note.—We may oftentimes obtain a near value of an assurance for a given *term* (particularly when only one life, A, is concerned) by the following simple method :—The series expressing such value is, by the corollary, equal to $\frac{s}{l_x}v(l_x-l_{x1})+v^2(l_{x1}-l_{x2})+v^3(l_{x2}-l_{x3})+\ldots v^m(l_{x+m-1}-l_{x+m})$: now, if we suppose the decrements of life to be equal and uniform throughout the given term (which is usually the case in the middle ages of life, particularly if the Northampton tables are made use of), and if such decrements be denoted by d; then will this series become $\frac{s}{l_x}(vd+v^2d+v^3d+\ldots v^m d)=\frac{sd}{l_x}\times\frac{1-v^m}{r}$. But, since d should in all those cases where the decrements are not *exactly* uniform be taken equal to $\frac{l_x-l_{x+m}}{l_m}$, this formula may be more conveniently expressed by $\frac{s(l_x-l_{x+m})}{m l_x}\,\frac{(1-v^m)}{r}$: whence the following rule :—

Divide the probability that the life shall fail in the given term by the number of years; multiply the quotient by the present value of an annuity for the given term, and also by the given sum: the product will be the value required.

It will readily appear that, if the given term is wholly within the limits of equal decrements, the above formula will give the *exact* value of the assurance for such term: and that if the decrements are not exactly uniform, but nearly so, it will not materially differ from the true value of such assurance. The formula, however, must always be used with proper caution, and with reference to the table of observations employed; as it is not suited to every case.

The same observations will apply to the case of *joint* lives, if d be taken equal to $\frac{l_x\,l_y\,l_z-l_x\,l_y\,l_{z|m}}{m}$; but we are not so likely, in practice, to meet with cases of assurances on joint lives for terms, where such terms fall within the limits of equal decrements.

For examples of the use and application of this corollary, see Question 29 in Chapter XII.

COROLLARY III.[1]

§ 185. Though in this problem the investigation is confined to the case of three joint lives, yet it is easy to apply the same method of reasoning to the case of any single life, or to any number of joint lives, or to the longest of any number of lives, or to any number of lives out of any other number of lives. And it will be found that by substituting the value of an annuity on such single life, on such joint lives, on such longest lives, &c. &c., instead of the quantity denoted by a_{xyz} in the given formula in § 179, we shall obtain the true values in such cases accordingly.

§ 186. The same remark will also apply to the two preceding corollaries, if (in addition to the substitution above alluded to) we also substitute the value of similar annuities Deferred for the given term, instead of the quantity $a_{xyz(m}$ in either formula in § 181 or § 183. But here it should be particularly observed that, when a Deferred or Temporary assurance depends on the *longest* of any number of lives, the quantity $\frac{v^m l_x\, l_y\, l_z\,{}_{|m}}{l_x\, l_y\, l_z}$ in the given formula will denote the expectation that the *longest* of such lives will receive £1 at the end of the given term: and that, when such assurance depends on the extinction of any *two out of three* lives, the same quantity will denote the expectation that any *two out of three* lives will receive £1 at the end of the given term, &c. &c.

For examples of the use and application of this corollary, see Questions 27, 28, and 29 in Chapter XII.

PROBLEM XXIII.[2]

§ 187. To find the value of an *annuity* to commence at the end of the year[3] in which any number of lives become extinct.

SOLUTION.

Let us, as in the preceding problem, confine the case to three joint lives A B C, whose probabilities of continuing 1, 2, 3, &c., years are as denoted in § 24; and let the annuity be a perpetuity, or estate in fee. Now, the chance which the heir of this estate has of receiving the rent at the end of any one year will depend altogether on the joint lives becoming extinct before the end of that year. The probability of this event happening before the end of the first year is $1-\frac{l_x\,l_y\,l_{z|1}}{l_x\,l_y\,l_z}$, which being multiplied by v will give the value of his expectation of receiving the first year's rent. In like manner, the probability of the joint lives failing before the end of the second year is $1-\frac{l_x\,l_y\,l_{z|2}}{l_x\,l_y\,l_z}$, which being multiplied by v^2 will give the value of his expectation of receiving the second year's rent. By a similar method of reasoning it will be found that $1-\frac{l_x\,l_y\,l_{z|3}}{l_x\,l_y\,l_z}$ multiplied by v^3 will give the value of his expectation of receiving the third year's rent; and so on, *ad infinitum.* For, since the estate must eventually revert to the heir, his expectation will never cease. Consequently the sum of all those values continued to infinity will be the total present value of the estate to be enjoyed after the extinction of the joint lives A B C: or, in other words, it will be the number of years' purchase which a person ought to give to have it assured to him after the extinction of those lives.

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[1] Simpson, Prob. 22, and *Sup.* Prob. 27. Morgan, Prob. 8, cor. and Prob. 9, cor.

[2] De Moivre, Prob. 6 and 29. Dodson, vol. ii. Ques. 88.

[3] That is, the *first* payment of the annuity is to be then made.

§ 188. But the sum of the above terms is equal to the following series:—

$$v+v^2+v^3+\ldots \textit{ ad infinitum}-\frac{1}{l_x\,l_y\,l_z}\,v\,l_x\,l_y\,l_{z|1}+v^2\,l_x\,l_y\,l_{z|2}+v^3\,l_x\,l_y\,l_{z|3},\ \text{\&c.}$$

The former of these is equal to $\frac{1}{r}$, or the present value of the perpetuity of an annuity of £1 per annum: but the latter, which is to be subtracted, will cease on the extinction of the oldest life, and is therefore equal to the value of an annuity on the joint lives ABC. Consequently the total value of the reversionary annuity becomes equal to $\frac{1}{r}-a_{xyz}$: and though this case is confined to that of three joint lives, yet it is easy to see that the method of solution will equally apply to that of any other number of lives; whence the following rule:—

§ 189. *Subtract the value of an annuity on the given lives from the present value of the estate or given annuity: the difference will be the value of the reversionary estate or annuity required.*

For examples of the use and application of this problem, see Question 26 in Chapter XII.

COROLLARY I.

§ 190. If the annuity, instead of being a perpetuity is only for a number of years ($=m$) *greater* however than that to which it is probable the given lives may extend,[1] we must substitute the value of such terminable annuity instead of the value of the perpetuity above mentioned: whereby the formula becomes $\frac{1-v^m}{r}-a_{xyz}$; and, consequently, the rule above given is also applicable to the present case.

For examples of the use and application of this corollary, see Question 26 in Chapter XII.

COROLLARY II.[2]

§ 191. But, if the annuity (instead of being a perpetuity, or for any long period) is for a term of years ($=m$) which is *less* than that to which it is probable the given lives may extend; the series, expressing the required value in the problem, will terminate at the end of that period, and will consequently become equal to $v+v^2+v^3+\ldots v^m-\frac{1}{l_x\,l_y\,l_z}(vl_x\,l_y\,l_{z\|1}+$ $v^2l_x\,l_y\,l_{z\|1}+v^3l_x\,l_y\,l_{z|3}+\ldots v^ml_x\,l_y\,l_{z|m}$. But, the first part is equal to $\frac{1-v^m}{r}$; and the latter part is, by Prob. I. cor. 4, equal to $-a_{xyz)m}$; whence the following rule:—

§ 192. *From the value of an annuity certain for the given term, subtract the value of a Temporary annuity on the given lives for the given term: the difference will be the value required.*

COROLLARY III.

§ 193. If the contingency (on which the payment of the annuity mentioned in the problem depends) is *Deferred* for a given number of years ($=m$); the series expressing the general value of the reversion of the perpetuity must commence at the end of that period, and be continued on to infinity; that is, the series $v^{m+1}(1-\frac{l_x\,l_y\,l_{z|m1}}{l_x\,l_y\,l_{z|m}})+v^{m+2}(1-\frac{l_x\,l_y\,l_{z|m2}}{l_x\,l_y\,l_{z|m}})+$ $v^{m+3}(\frac{l_x\,l_y\,l_{z|m3}}{l_x\,l_y\,l_{z\|m}})$......*ad infinitum*, will denote the value of the required reversionary annuity, to be entered on provided either of the given lives become extinct after the *m*th year. Now this series may be resolved into two others, namely, $v^m(v+v^2+v^3 \ldots\ldots \textit{ad infinitum})-\frac{v^m}{l_x\,l_y\,l_{z|m}}(vl_x l_y l_{z|+1m}+v^2 l_x\,l_y\,l_{z|2+m}+v^3 l_x\,l_y\,l_{z|3+m}+\ldots$ the first of which is equal to $\frac{v^m}{r}$; and the latter is, by Prob. I. cor. 3, equal to $-v^m a_{xyz|m}$. Consequently, the value of the reversionary annuity would be equal to $v^m(\frac{1}{r}-a_{xyz|m})$, were it certain that the given lives would continue in being to the end of m years; but as the probability of this event is $\frac{l_x\,l_y\,l_{z|m}}{l_x\,l_y\,l_z}$, we must multiply the above expression by this quantity, in order to obtain the total present value of the same. Whence the true value will be equal to $\frac{v^m\,l_x l_y\,l_{z|m}}{rl_x\,l_y\,l_z}-a_{xyz|m}(\frac{v^m l_x\,l_y\,l_{z|m}}{l_x\,l_y\,l_z}$. But $a_{xyz|m}\frac{v^m l_x\,l_y\,l_{z|m}}{l_x\,l_y\,l_z}$ is, by Prob. I. cor. 3, equal to $a_{xyz(m}$; therefore the above value becomes $\frac{v^m l_x l_y l_{z|m}}{rl_x\,l_y\,l_z}-a_{xyz(m}$: and whence the following rule:[1]—

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[1] The term to which it is probable that any given life or lives may extend is—for a *single* life, equal to the difference between the age of such life and the age of the oldest life in the table of observations: for *joint* lives, equal to the difference between the *oldest* of such lives and the age of the oldest life in the table:—for the *longest* of any lives, equal to the difference between the age of the *youngest* of such lives and the age of the oldest life in the table.

[2] Simpson's *Sup.* Prob. 14. De Moivre, Prob. 28. Morgan, Prob. 4, cor. 2. Dodson, vol. ii. Ques. 92.

§ 194. *Multiply the expectation of the given lives receiving £1 at the end of the given term by the perpetuity; from the products subtract the value of an annuity on the given lives Deferred for the given term: the difference will be the value required.*

COROLLARY IV.[2]

§ 195. Or, if the contingency is *Temporary*, that is, if it be required to ascertain the value of such annuities depending on the extinction of any given lives, provided that shall happen *within a given period* $(=m)$; the value of the assurance will be expressed by the series $v(1-\frac{l_x\,l_y\,l_{z|1}}{l_x\,l_y\,l_z})+v^2(1-\frac{l_x\,l_y\,l_{z|2}}{l_x\,l_y\,l_z})+v^3(1-\frac{l_x\,l_y\,l_{z|3}}{l_x\,l_y\,l_z})+\ldots v^m(1-\frac{l_x\,l_y\,l_{z|m}}{l_x\,l_y\,l_z})+v^m{}_+{}^1(1-\frac{l_x\,l_y\,l_{z|m}}{l_x\,l_y\,l_z})+v^m{}_+{}^2(1-\frac{l_x\,l_y\,l_{z|m}}{l_x\,l_y\,l_z})+$ *ad infinitum.* For, the mth year's rent *and all the subsequent ones* being dependent on the given lives becoming extinct in m years, it is obvious that all such subsequent ones must be multiplied by the common factor $(1-\frac{l_x\,l_y\,l_{z|m}}{l_x\,l_y\,l_z})$. But the first m terms of this series are equal to $\frac{1-v^m}{r}-a_{xyz)m}$; as found by the second corollary; and the remaining terms are equal to $v^m(1-\frac{l_x\,l_y\,l_{z|m}}{l_x\,l_y\,l_z})(v+v^2+v^3$ ...... *ad infinitum*$)$ $=$ $v^m(1-\frac{l_x\,l_y\,l_{z|m}}{rl_x\,l_y\,l_z})$. Consequently the total value of the series becomes $\frac{1-v^m}{r}-a_{xyz)m}+\frac{v^m}{r}-\frac{v^m l_x\,l_y\,l_{z|m}}{rl_x\,l_y\,l_z}$; which, since $(a_{xyz)m}=a_{xyz}-a_{xyz(m})$ may be further reduced to $\frac{1}{r}-a_{xyz}-[\frac{v^m l_x\,l_y\,l_{z|m}}{rl_x\,l_y\,l_z}-a_{xyz(m}]$; and whence the following rule:[1]—

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[1] See the note in p. 96.

[2] Simpson, Prob. 23; and *Sup.* Prob. 31. Price, Note (G). Dodson, vol. ii. Ques. 93.

§ 196. *From the whole value of the reversion of the perpetuity after the given lives, subtract the value of the same Deferred for the given term: the difference will be the value required.*

COROLLARY V.

§ 197. It will be obvious, from what has been said in the first corollary, that if the annuity, instead of being a perpetuity, is for a given term only, the rules given in the two preceding corollaries will still be correct, provided we substitute, for the *perpetuity*, the value of an annuity for the *given term*. Such terminable annuity, however, must always be for a term of years *greater* than that for which the assurance is made.

COROLLARY VI.

§ 198. Though the above problem and its corollaries are confined, in the investigation, to the case of three joint lives, yet it is easy to apply the same method of reasoning to the case of any single life, or to any number of joint lives, or to the longest of any number of lives,[2] or to any number of lives out of any other number of lives, &c. &c., either for the *whole* lives or for a given *term*.[3] And it will be found that by substituting the value of an annuity on such single life, on such joint lives, on the longest of any number of lives, &c. (either for the whole life or Deferred for the given term), instead of the value a_{xyz} or $a_{xyz(m}$ in the given formulæ, we shall obtain the true values in such cases accordingly.

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[1] If the annuity is to be entered upon *at the end of the given time* (in case of the failure of the given lives), and not *at the end of the year in which such failure may happen*, its present value will be equal to

$$v^m(1-\frac{l_x\,l_y\,l_{z|m}}{l_x\,l_y\,l_z})(1+\frac{1}{r})=\frac{v^{m-1}}{r}(1-\frac{l_x\,l_y\,l_{z|m}}{l_x\,l_y\,l_z}):$$

for, such reversionary perpetuity is evidently to be entered on and enjoyed at the end of the *m*th year, provided the given lives fail prior thereto. And care should be taken not to confound this case with the one mentioned in the text.

[2] Dodson, vol. ii. Ques. 90, 91.

[3] Simpson, Prob. 24.

§ 199. It should be particularly observed, however, that when the *longest* of any number of lives are concerned, the quantity $\frac{v^m l_x l_y l_z \,|m}{l_x l_y l_z}$ will represent the expectation that the *longest* of such lives will receive £1 at the end of the given term. And a similar remark will apply when the assurance depends on the extinction of any *two out of three* lives, &c., &c. See Problem XXII. cor. 3.

SCHOLIUM.

§ 200. From these two problems and their corollaries may be determined all questions relative to the value of assurances of any given sum, or of any given annuity, depending on the contingencies therein stated; and as any given sum may be converted into a corresponding perpetuity (or into a perpetual annuity of a corresponding value) by multiplying it by the interest of £1 for a year, it would seem at first to be the same thing to determine the present value of a given *sum* depending on the extinction of any lives, as to determine the value of a corresponding *annuity* for ever, depending on the same lives: that is, it would appear to be the same thing to determine the present value of £100, payable at the end of the year in which any given lives became extinct, as to determine the value of an estate yielding £5 per annum, and to be entered upon at the same period; interest being reckoned at 5 per cent. But, it should be observed, that the first yearly payment of a reversionary *annuity* becomes due and is payable at the end of the year in which the lives fail; however much or little of that year may then happen to be unexpired: and this likewise is the time when a reversionary *sum* becomes due.. The expectant, therefore, in the former case will have entered on his annuity, or received the first year's rent of it, at the very time that the expectant of the sum is supposed to have laid out such sum in the purchase of a perpetuity of a corresponding value; the first year's rent of which he will not receive till the *end of the following year*. Consequently, a reversionary estate is worth one year's purchase more than a corresponding reversionary sum: whence the former is to the latter in the ratio of £1, increased by its interest for one year, to £1; that is, as $(1+r)$ to 1.

§ 201. Therefore, if the present value of a reversionary sum be multiplied by $(1+r)$ it will give the value of a corresponding reversionary estate or annuity. Thus, the present value of a reversionary annuity of £1 per annum for ever after the extinction of the joint lives A B C, is, by Prob. XXIII., equal to $\frac{1-ra_{xyz}}{r}$; and the present value of a corresponding reversionary sum $(=\frac{1}{r})$ after the same lives, is by Prob. XXIII. equal tc

$\frac{v(1-ra_{xyz})}{r}$; but it is evident on inspection that the former value is to the latter in the ratio of $(1+r)$ to 1.

§ 202. Hence it is evident also that the value of any reversionary annuity after any given lives, being divided by the amount of £1 in a year, will give the present value of a corresponding reversionary sum after the same lives.[1]

These remarks may be more fully confirmed by any of the similar examples in the two problems, or their corollaries, as may be seen in the Scholium to Question 27 in Chapter XII.

CHAPTER VII.

ON SUCCESSIVE LIFE ANNUITIES AND COPYHOLD ESTATES.

§ 203. In all the preceding problems, involving the question of reversionary life annuities, the lives of the expectants have been supposed to be such as are *now* fixed on and determined; and the value of an annuity on their lives consequently becomes less and less, according as their period of coming into possession might be prolonged. In such questions, however, as relate to the present division of the subject, the life which is to succeed to the annuity, after the extinction of the life in possession, is supposed to be one which is *then* to be fixed on at pleasure; and which will probably be one of the best lives that can then be found. This life, therefore, may be considered as having a fixed and determinate value, since such a life may generally be chosen as will best answer the views of persons concerned in questions of this kind; and it is usual to conceive a *mean* age at which they are all admitted. But the nature of the cases, which involve the consideration of this subject, will best appear from the following problems:—

PROBLEM XXIV.[2]

§ 204. Supposing A to enjoy an annuity for his life, and, at his decease, to have the nomination of a successor, B, who is also to enjoy the annuity for his life: To find the present value of the annuity on the succeeding life, and also the value of the two successive lives.

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[1] Mr. Simpson not having attended to this circumstance, it becomes necessary to correct the rules given by him for the solution of Probs. XXI. and XXII. in his *Doctrine of Annuities*, &c.; and of Probs. XXVI., XXVII., XXXII., XXXIII., and others of a similar kind, in his *Supplement*. The same observation will apply to the problems of M. De Moivre and Mr. Dodson, alluded to in the note in p. 94.

[2] Simpson, Prob. 25, and *Sup*. Prob. 24. De Moivre, Prob. 13. Morgan, Prob. 14.

SOLUTION.

Let the succeeding life B, to be put in nomination at the decease of A, be such that an annuity on his life *at that time* may be equal to a_y. Now, since the probability that the first life fails, or that the second comes into possession, in the first year is $\frac{d_x}{l_x}$; and as the total value of what the second life will be entitled to, on the happening of this event is a_y, it follows that his expectation of coming into possession the first year (or of receiving an equivalent sum, equal to a_y,) will, as in Prob. XXII., be denoted by $\frac{a_y v d_x}{l_x}$. In like manner, it will be found that his expectation of coming into possession in the second year (or of receiving an equivalent sum a_y) will be denoted by $\frac{a_y v^2 d_{x1}}{l_x}$; and that his expectation of coming into possession in the third year (or of receiving the equivalent sum a_y) will be denoted by $\frac{a_y v^3 d_{x2}}{l_x}$, and so on for every succeeding year to the utmost extent of A's life.

But the sum of all these values is equal to $\frac{v a_y}{l_x}(l_x + v l_{x1} + v^2 l_{x2} + v^3 l_{x3} + \ldots) - \frac{a_y}{l_x}(v l_{x1} + v^2 l_{x2} + v^3 l_{x3} + \ldots)$; which, by Prob. XXII., is equal to $a_y v(1 - r a_x)$; and which would be the value required, were a_y in reality the value of a reversionary *sum* to be received on the decease of A: but since it denotes the value of an *annuity*, the first payment of which commences at the end of the year in which the life A fails, we must multiply the above expression by $(1+r)$, agreeably to what has been said in the scholium in § 200; whence, the true present value of the successive life will be $a_y(1 - r a_x)$; and whence the following rule:—

§ 205. *Multiply the value of an annuity on the life in possession by the rate of interest, and subtract the product from unity; multiply the remainder by the assumed value of an annuity on the succeeding life: the product will be the present value of an annuity on such succeeding life.*

§ 206. If this present value be added to the value of an annuity on the life in possession, it will give $a_x + a_y(1 - r a_x)$ for the value of the *two* successive lives.

For examples of the use and application of this problem, see Question 23 in Chapter XII.

COROLLARY I.

§ 207. Hence may be determined the present value of an annuity on any number of *joint* lives, or on the *longest* of any number of lives, &c., nominated to succeed after any other number of joint lives, or after the longest of any other number of lives, &c.: for, by making a_x and a_y, respectively, equal to the value of an annuity on such joint lives, or the longest of such lives, &c., the above formula will express the true present values of such successive lives.

COROLLARY II.[1]

§ 208. If the succeeding life, instead of receiving an annuity *during his life*, were to receive an annuity *certain for a given term of years* after the failure of the life in possession; then, by making a_y equal to the value of such an annuity certain for the given term, the above formula would truly express the present value of such annuity to be entered on at the failure of the life in possession.

And if this annuity were a perpetuity (that is, if the succeeding life and his heirs were to receive an annuity for ever after the failure of the life in possession), a_y would become equal to $\frac{1}{r}$, and the formula would in this case become equal to $\frac{1}{r} - a_x$; which is the very same as that deduced from Prob. XXIII.

For examples of the use and application of this corollary see Question 24 in Chapter XII.

PROBLEM XXV.[2]

§ 209. Three lives, A, B, C, being given in succession: To find the present value of an annuity on the third succeeding life; and also the value of the three successive lives.

SOLUTION.

Let the values of an annuity on each of the three lives, at the time that they severally come into possession, be respectively denoted by a_x, a_y, a_z. Therefore, since the value of an annuity on the second life in succession (to commence at the decease of A) is to the value of a perpetuity (to commence at the same time) in the ratio of a_y to $\frac{1}{r}$; it follows that the present value of the former will be to the present value of the latter in the same ratio. But, the present value of the latter is, by Prob. XXIII., equal to $\frac{1}{r}-a_x=\frac{1-ra_x}{r}$: therefore $\frac{1}{r}: a_y=\frac{1-ra_x}{r}: a_y(1-ra_x)=$ the present value of the former; and which is the same value as that found by the last problem: whence the two methods of solution confirm the truth of each other.

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[1] Morgan, Prob. 13.

[2] Simpson, Prob. 26, and *Sup.* Prob. 25.

The present value of the first two successive lives being thus found equal to $a_x+a_y(1-ra_x)$; it follows that the present value of the reversion of a perpetuity after these lives will (by a deduction from Prob. XXIII.) be equal to $\frac{1}{r}-[a_x+a_y(1-ra_x)]=\frac{1}{r}(1-ra_x)(1-ra_y)$. Consequently, since the value of an annuity on the third life in succession (to commence at the decease of B) is to the value of a perpetuity (to commence at the same period) in the ratio of a_z to $\frac{1}{r}$; it follows that the present value of the former will be to the present value of the latter, in the same ratio: that is, $\frac{1}{r}: a_z=\frac{1}{r}(1-ra_x)(1-ra_y): a_z(1-ra_x)(1-ra_y)=$ the present value of an annuity on the third successive life: whence the following rule.

§ 210. *Multiply the value of an annuity on the life in possession by the rate of interest, and subtract the product from unity; multiply also the assumed value of an annuity on the second life in succession by the rate of interest, and subtract this product likewise from unity: multiply together these two remainders, and their product again by the assumed value of an annuity on the third life in succession; this last product will be the value of the third successive life.*

§ 211. If the present value of each successive life, as above found, be added together, their sum, or $a_x+a_y(1-ra_x)+a_z(1-ra_x)(1-ra_y)$, will be the present value of the three successive lives: but this expression will be found equal to $\frac{1}{r}[1-(1-ra_x)(1-ra_y)(1-ra_z)]$: whence the present value of any number of lives in succession may be discovered on inspection; and thence the following rule:—

212. *Multiply the assumed value of an annuity on each of the proposed lives, by the rate of interest; take the several products from unity, and multiply together all the remainders; let the product thus arising be also subtracted from unity, and the remainder divided by the rate of in-*

terest: the quotient will be the present value of all the successive lives, including the life in possession.

PROBLEM XXVI.[1]

§ 213. Suppose a person to purchase a copyhold estate, on any number of lives, A, B, C, &c., for the sum s, on condition that he and his successors may renew it continually by paying the fine f whenever any one of such lives becomes extinct: To find the present value of the whole purchase of such estate.

SOLUTION.

Let the value of an annuity on each of the lives A, B, C, &c., in possession, be respectively denoted by a_x, a_y, a_z, &c.: and let the value of an annuity on each of the lives A_1, A_2, A_3, etc. (which are supposed to follow in direct succession from A), be respectively denoted by a_1, a_2, a_3, &c.[2] In like manner, let the value of an annuity on each of the lives B_1, B_2, B_3, &c. (which are supposed to follow in direct succession from B), be respectively denoted by b_1, b_2, b_3, &c.: and so on with respect to the lives immediately succeeding, C, D, &c. And let us first determine the present value of all the fines payable on the extinction of the life A and his immediate successors.

Now, the present value of the fine f, payable on the decease of A, in whatever part of the year that may happen, may in the present case be considered equal to the present value of an estate, yielding fr per annum, to be entered upon at the decease of A; which, by Prob. XXIII., is found to be $fr\ (\frac{1}{r}-a_x)=f(1-ra_x)$. If, instead of a_x in this formula, we substitute the present value of the two successive lives A, A_1, which, by Problem XXIV., is found equal to $a_x+a_1(1-ra_x)$, we shall have $f(1-ra_x)(1-ra_1)$ for the present value of the fine to be paid on the decease of A_1. And if, instead of a_x in that same formula, we substitute the present value of the three successive lives, A, A_1, A_2, which, by the last problem, is found equal to $a_x+a_1(1-ra_x)+a_2(1-ra_x)(1-ra_1)$, we shall have $f(1-ra_x)(1-ra_1)(1-ra_2)$ for the present value of the fine payable on the decease of A_2: and so on with respect to all the subsequent fines payable on the extinction of each life in direct succession from

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[1] Simpson, Prob. 27; and *Sup.* Prob. 29. Dodson, vol. iii. Ques. 82 to 95. De Moivre, Prob. 10.

[2] Care must be taken not to mistake these numeral quantities for the ages of the lives to which they are annexed, as they merely denote the order of succession among the given lives.— EDITOR.

A. But the sum of all these quantities, or the series $f[(1-ra_x)+(1-ra_x)(1=ra_1)+(1-ra_x)(1-ra_1)(1-ra_2)+ \ldots\ldots$ *ad infinitum*] is the present value of all the sums that may be paid from time to time for the renewals of the several lives in direct succession from A.

By a similar method of proceeding it will be found that the series $f[(1-ra_y)+(1-ra_y)(1-rb_1)+(1-ra_y)(1-rb_1)(1-rb_2+\ldots\ldots$ *ad infinitum*] will denote the present value of all the sums that may be paid from time to time for the renewals of the several lives in direct succession from B. And so on with respect to the other lives, C, D, &c., and their successors for ever; and the sum of all these different series, or

$$\begin{array}{c}
f[(1-ra_x)+(1-ra_x)(1-ra_1)+\ldots\ldots \textit{ad inf.}]+ \\
f[(1-ra_y)+(1-ra_y)(1-rb_1)+\ldots\ldots \textit{ad inf.}]+ \\
f[(1-ra_z)+(1-ra_z)(1-rc_1)+\ldots\ldots \textit{ad inf.}]+ \\
\&c. \qquad \&c. \qquad \&c.
\end{array}$$

will be the total present value of all the fines that the tenant can ever pay; and which, being added to s, will give the whole value paid for the purchase.

COROLLARY I.

§ 214. Hence, if the lives with which the lease is from time to time renewed, be supposed equal to one another, or of the same common age A_1, the general expression above given will become

$$\begin{array}{c}
f(1-ra_x)[1+(1-ra_1)+(1-ra_1)^2+(1-ra_1)^3+\ldots\ldots \textit{ad inf.}]+ \\
f(1-ra_y)[1+(1-ra_1)+(1-ra_1)^2+(1-ra_1)^3+\ldots\ldots \textit{ad inf.}]+ \\
f(1-ra_z)[1+(1-ra_1)+(1-ra_1)^2+(1-ra_1)^3+\ldots\ldots \textit{ad inf.}]+ \\
\&c. \qquad \&c. \qquad \&c.
\end{array}$$

the sum of which series, since $[1+(1-ra_1)+(1-ra_1)^2+(1-ra_1)^3+\ldots\ldots$ *ad infinitum*] is equal to $\frac{1}{ra_1}$,[1] becomes $\frac{f}{a_1}(\frac{1}{r}-a_x)(\frac{1}{r}-a_y+\frac{1}{r}-a_z+\ldots)=\frac{f}{a_1}(\frac{n}{r}-a_x-a_y-a_z,$ &c.); where n denotes the number of lives on which the estate is held, whence the following rule :—

§ 215. *Divide the number of lives by the rate of interest, and from the quotient subtract the sum of the values of an annuity on each of the single lives in possession; divide the remainder by the assumed value of an annuity on the common life with which the lease is from time to time to be renewed: the quotient, thence arising, multiplied by the fine to be paid on renewing, will be the total present value of all the renewals for ever; and*

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[1] It is well known that $\frac{1}{1-x}=1+x+x^2+x^3+\ldots\ldots$ *ad infinitum:* therefore, by substituting $(1-ra_1)$ for x, we shall have $\frac{1}{ra_1}$, equal to the series given in the text.

which being added to the sum given for the estate, will give the whole value of the purchase.

§ 216. *Example.* Suppose a person to have paid down £1000 for the purchase of a copyhold estate held on three lives, whose ages are 30, 50, and 70; on condition that he may, on the extinction of any life, continually renew with any other life that he thinks proper, on paying a fine of £600: What is the present value of all those fines with which the estate may be continually renewed, reckoning interest at 4½ per cent., and the probabilities of living according to the observations of *M. De Parcieux*?

Here we shall have $a_x = 15{\cdot}691$, $a_y = 11{\cdot}921$, $a_z = 6{\cdot}221$, $n = 3$, $r = {\cdot}045$, $f = 600$, and A_1 (or the value of an annuity on the best life in the table[1]) $= 17{\cdot}515$. Consequently $\frac{600}{17{\cdot}515}(60 - 33{\cdot}833) = 896{\cdot}386$, or £896, 7s. 9d. will be the present value of all the fines, and which being added to the £1000 paid upon entering will give the total value of the fee-simple of the estate.

COROLLARY II.

§ 217. When all the lives in possession are of the same common age A, the formula in the preceding corollary will become equal to $\frac{fn}{a_1}(\frac{1}{r} - a_x)$. But if all the lives, as well those in possession as those to be put in nomination afterwards, be equal to each other, or of the same common age A_1, the present value of all the renewals for ever will then be equal to $\frac{fn}{a_1}(\frac{1}{r} - a_1) = fn(\frac{1}{ra_1} - 1)$.

COROLLARY III.

§ 218. If the present value found by either of the preceding corollaries be multiplied by the rate of interest, it will show how much the rent-roll of the landlord's estate ought in each case to be increased on account of the fines paid at renewing.

Thus, in the example given in § 216, it will be found that $896{\cdot}386 \times {\cdot}045 = 40{\cdot}337$, or £40, 6s. 9d. is the sum by which the rent-roll of the lord's estate ought to be increased on account of the fines there mentioned.

COROLLARY IV.

§ 219. Since the purchase money paid for the lease, together with the present value of all the fines to be paid on renewal, is equal to the value

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[1] This is always the assumed value of A_1, agreeably to what has been said on this subject in § 203.

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of the perpetuity of the rack-rent of the estate $(=p)$; that is, since $s+\frac{f}{a_1}(\frac{n}{r}-a_x-a_y-a_z-\&c.)=p$; it follows that f, or the fine which ought to be paid on renewing, will be equal to $\frac{(p-s)a_1}{\frac{n}{r}-a_x-a_y-a_z-\&c.}$: whence the following rule:—

§ 220. *Subtract the tenant's interest in the lease (or the purchase money which he has given for the same) from the value of the fee-simple of the estate, and multiply the remainder by the assumed value of an annuity on the common life, with which the lease is supposed to be constantly renewed; reserving the product: divide the number of lives on which the lease is now held, by the rate of interest; and from the quotient subtract the sum of the values of an annuity on each of those lives: the reserved product being divided by this remainder, will give the sum which ought in justice to be paid as a fine on each renewal.*

§ 221. *Example.* Suppose a person to have purchased, for £1000, a copyhold estate, the rack-rent of which is estimated at £100 per annum; and that such estate is held on three lives, renewable for ever on the extinction of either of those lives, by paying a fine certain: What ought such a fine be fixed at, in order that the purchaser may make 5 per cent. interest of his money, supposing the ages of the lives (on which the estate is now held) to be 30, 50, and 70, and that the probabilities of living are according to the observations of *M. De Parcieux*?

Here we shall have p (or the value of the perpetuity of the rack-rent of the estate)$=2000$, $s=1000$, and the remaining quantities as in the example in § 216. Consequently the value of the fine ought to be $\frac{1000\times17{\cdot}515}{60-33{\cdot}833}=669{\cdot}354$, or £669, 7s. 1d.

COROLLARY V.

§ 222. If the estate is held on one life only, the present value of the landlord's interest therein will be *universally* expressed by $f\times\frac{1-ra_x}{ra_1}$. Now, *immediately after* the receipt of a fine, the life in possession is equal to A_1; whence the expression in this case becomes $f\times\frac{1-ra_1}{ra_1}$: and *immediately before* the receipt of a fine, the life in possession having become extinct, the expression in this case becomes $\frac{f}{ra_1}$.

Note.—These three formula will serve to express the value of perpetual

Advowsons (considered as an object of traffic) under the three most usual circumstances: (1.) where the living is *possessed by an incumbent* whose life is equal to A; (2.) immediately *after presentation*, where the life presented is of the same common age, A_1, as that with which the living is supposed to be constantly filled up; (3.) with *immediate resignation*.

It must however be here particularly observed, that in *these* cases the value of a_1 (or the value of an annuity on the common life with which the living is supposed to be constantly filled up) must never be assumed so great as in those cases mentioned in the text: because the person, who is presented to the living, must always be above 24 years of age; and it seldom happens that he is even so young as this.

It has been ingeniously suggested that the ages of the incumbents, when they are inducted, may be partly fixed from the value of the livings. See De Moivre, Prob. XIV. and XV.; and Dodson, vol. iii. p. 347.

But since the present value of the next fine is universally expressed by $f(1-ra_x)$, or by the amount of such fine multiplied into the difference between unity and the product of the rate of interest by the value of an annuity on the life in possession, we may readily determine the landlord's interest in the estate, or the value of all the fines to be paid on renewing, by the following rule:—

§ 223. *Divide the present value of the next fine by the product of the rate of interest into the value of an annuity on the common life with which the lease is to be continually renewed: the quotient thence arising will be the value required.*

§ 224. *Example.* Suppose that a copyhold tenant pays to the lord of the manor a fine of £100 on his admission, and that every successor does the same; what is the present value of the lord's interest in that copyhold, on the supposition that the tenants admitted thereto are (one with another) 25 years of age at the time of their admission: interest being reckoned at 4 per cent., and the probabilities of living as at *Northampton?*

Here we shall have $a_1=15{\cdot}438$, $r={\cdot}04$, and $f=100$; consequently the value of the lord's interest *immediately before* the receipt of a fine will be equal to $100\times\frac{1}{{\cdot}04\times15{\cdot}438}=161{\cdot}938$, or £161, 18s. 9d.; and *immediately after* the receipt of a fine it will be equal to $100\times\frac{1-{\cdot}04\times15{\cdot}438}{{\cdot}04\times15{\cdot}438}$ $=61{\cdot}935$, or £61, 18s. 9d. But if the life now in possession be 70 years of age, we shall have $a_x=6{\cdot}361$; in which case the lord's interest will be £$100\times\frac{1-{\cdot}04\times6{\cdot}361}{{\cdot}04\times15{\cdot}438}=113{\cdot}174$, or £113, 3s. 6d.

Therefore if the tenant gave £500 for the lease, the whole value of the purchase may in this latter case be estimated at £613, 3s. 6d., and the corresponding rent at £24, 10s. 6d.

COROLLARY VI.[1]

§ 225. If the estate is held on the *longest* of any number of lives (that is, on condition that, whenever *all* those lives become extinct, the lease may be renewed with the same number of lives, and on the same conditions, by paying the given fine) the formulæ in the last corollary, since all the lives may in this case be considered but as one, will still express the true value of the landlord's interest; if we make a_x denote the value of an annuity on the *longest* of all the lives in possession, and a_1 the assumed value of an annuity on the *longest* of all the lives with which the lease is to be continually filled up.

§ 226. *Example.* Suppose an estate to be leased on two lives, with condition that, on the extinction of both those lives, the same may be renewed with two other lives (the best that can be found) on paying a fine of £300; and so on for ever: What is the present value of the landlord's interest in the estate, taking the probabilities of living as at *Northampton*, and the rate of interest at 5 per cent.?

Here we shall have a_1 (or the value of an annuity on the longest of two lives, both aged 8 years) $=17{\cdot}721$, $r={\cdot}05$, and $f=300$; consequently the value of the landlord's interest *immediately before* the receipt of a fine will be equal to $300\times\frac{1}{{\cdot}05\times17{\cdot}721}=340{\cdot}656$, or £340, 13s. 1d.; and *immediately after* the receipt of a fine it will be equal to 40·656. But if the ages of the lives, on which the estate is now held, be 40 and 60 years of age, the landlord's interest will be equal to $300\times\frac{1-{\cdot}05\times13{\cdot}214}{{\cdot}05\times17{\cdot}721}$ $=114{\cdot}880$; or, if the eldest of those lives be extinct, the landlord's interest will be equal to $300\times\frac{1-{\cdot}05\times11{\cdot}837}{{\cdot}05\times17{\cdot}721}=138{\cdot}192$.

SCHOLIUM.

§ 227. From the principles here laid down, it will be easy to determine whether it is most advantageous, to the lessee or the landlord, to fill up a life as soon as it becomes vacant, or to wait till two or more of them have dropt before the renewal.

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[1] Simpson's *Sup.* Prob. 28. De Moivre, Prob. 12. Dodson, vol. iii. Ques. 85.

CHAPTER VIII.

ON ASSURANCES DEPENDING ON A PARTICULAR ORDER OF SURVIVORSHIP.

§ 228. The subject of the present chapter is certainly one of the most intricate in the whole doctrine of annuities, since it involves contingencies for which it is very difficult to give a concise and accurate expression. When two lives only are concerned, the investigations are not very complex, and the solutions may be obtained without much labour or inconvenience; but when three or more lives are involved in the question, the investigations become more intricate, and in many cases indeed baffle all our endeavours to obtain the correct value. These latter cases, which are equally numerous with those whose values we can obtain correctly, arise out of the subject already mentioned in page 82, and will mostly occur towards the end of this chapter. We may, indeed, approximate to their true value by the help of the two lemmata given in the fifth chapter, as will more distinctly appear hereafter.

I would here observe that I have not considered any cases where *more* than three lives are involved: those cases are so very rare that it would not be worth while to lay down any general rules on the subject; and to investigate them properly would swell the present work to an enormous bulk.

In order to avoid any unnecessary repetitions in the ensuing problems, I will take this opportunity of mentioning, once for all, that I shall in every case denote the given sum by s; and that the probabilities of living will be still represented by the same quantities as in § 23. The resulting formulæ, which show the value of such sum, will sufficiently enable the experienced analyst to determine its numerical value; but they are too complex and intricate to be inserted as rules, in words at length.

§ 229. I would also observe here, that I use the characters a_{x1}, a_{y1}, a_{z1}, to denote the value of an annuity on a life one year *older* than the life A, B, or C respectively: and the characters a_{1x}, a_{1y}, a_{1z} to denote the value of an annuity on a life one year *younger* than the life A, B, or C respectively. Consequently, when the character a_{x+m1}, a_{y+m1}, or a_{z+m1} occurs, it is meant to denote the value of an annuity on a life $(m+1)$ years older than A, B, or C respectively.

The same observations will apply to the characters a_{x+1m}, a_{y+1m}, or a_{z+1m}, which respectively denote the value of an annuity on a life $(m-1)$ older than A, B, or C respectively. This remark will also extend to the case of such lives considered jointly with any other lives; thus $a_{x1\cdot y}$ denotes the value of an annuity on the joint lives of B and a life one year older than A; $a_{x1\cdot yz}$ the value of an annuity on the joint lives of B C and a life one year older than A; and so on.

PROBLEM XXVII.[1]

§ 230. To determine the present value of a given sum, payable on the decease of A, provided that shall be the *first* which fails of two given lives, A, B.

SOLUTION.

The chance of receiving the sum at the end of any one year will depend on the happening of one or other of these two events: (1.) that A dies *in* the year, and that B lives to the end of it; (2.) that both lives fail *in* the year, restrained however to the contingency that A dies first. The probability that the first event will happen in the first year is $\frac{d_x\, l_{y1}}{l_x\, l_y}$; and the probability that the second event will happen in the same period is $\frac{d_x\, d_y}{2l_x\, l_y}$: these two values, therefore, being added together, and multiplied by sv, or the present value of the given sum *certain* to be received at the end of the year, will give $\frac{sv}{2}\left(\frac{l_x\, l_y}{l_x\, l_y}-\frac{l_{x1}\, l_{y1}}{l_x\, l_y}-\frac{l_{x1}\, l_y}{l_x\, l_y}+\frac{l_x\, l_{y1}}{l_x\, l_y}\right)$ for the value of the expectation of receiving such sum at the end of the first year. In like manner, since the probability that the first event will happen *in* the second year is $\frac{d_{x1}\, l_{x2}}{l_x\, l_y}$, and the probability that the second event will happen *in* the same period is $\frac{d_{x1}\, d_{y1}}{2l_x\, l_y}$, it follows that the sum of these, multiplied by sv, will give the value of the expectation of receiving the sum at the end of the second year. By a similar method of reasoning we may find the value of the expectation of receiving the sum at the end of the third year; and so on for every succeeding year, to the utmost extent of human life; the sum of all which yearly values, or the series

$$\begin{aligned}
&\frac{sv}{2}\left[\frac{l_x\, l_y}{l_x\, l_y}-\frac{l_{x1}\, l_{y1}}{l_x\, l_y}-\frac{l_{x1}\, l_y}{l_x\, l_y}+\frac{l_x\, l_{y1}}{l_x\, l_y}\right]+\\
&\frac{sv^2}{2}\left[\frac{l_{x1}\, l_{y1}}{l_x\, l_y}-\frac{l_{x2}\, l_{y2}}{l_x\, l_y}-\frac{l_{x2}\, l_{y1}}{l_x\, l_y}+\frac{l_{x1}\, l_{y2}}{l_x\, l_y}\right]+\\
&sv^3\left[\frac{l_{x2}\, l_{y2}}{l_x\, l_y}-\frac{l_{x3}\, l_{y3}}{l_x\, l_y}-\frac{l_{x3}\, l_{y2}}{l_x\, l_y}+\frac{l_{x2}\, l_{y3}}{l_x\, l_y}\right]+\\
&\qquad\&c. \qquad\qquad \&c. \qquad\qquad \&c.
\end{aligned}$$

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[1] Price, Ques. 11, and Note (M). Dodson, vol. iii. Ques. 23. Simpson's *Sup.* Prob. 32. Morgan, Prob. 16, and in *Phil. Trans.* for 1788, Prob. 2.

will be the total present value of s the sum to be received on the above contingency.

§ 231. But the sum of the first two of these perpendicular series (independent of the common multiplier $\frac{s}{2}$) is, by Prob. XXII., equal to $v(1-ra_{xy})$; the third will be found equal to $-v(1+a_{x1\cdot y})\,\frac{l_{x1}}{l_x}$; and the last equal to $\frac{a_{1x\cdot y}\,l_{1x}}{l_x}$. Consequently, the total present value of the given sum will be equal to $\frac{s}{2}\left[v(1-ra_{xy})-\frac{v(1+a_{x1\cdot y})\,l_{x1}-a_{1x\cdot y}\,l_{1x}}{l_x}\right]$.

Note.—The third of these perpendicular series, or $\frac{vl_{x1}\,l_y}{l_x\,l_y}+\frac{v^2l_{x2}\,l_{y1}}{l_x\,l_y}+\frac{v^3l_{x3}\,l_{y2}}{l_x\,l_y}\ldots$ (by taking all the terms as affirmative and omitting the common multiplier $\frac{s}{2}$), is evidently equal to $\frac{vl_{x1}\,l_y}{l_x\,l_y}\left[1+\frac{vl_{x2}\,l_{y1}}{l_{x1}\,l_y}+\frac{v^2l_{x3}\,l_{y2}}{l_{x1}\,l_y}+\frac{v^3l_{x4}\,l_{y3}}{l_{x1}\,l_y}+\ldots\right]$: which (since $\frac{vl_{x2}\,l_y}{l_{x1}\,l_y}+\frac{v^2l_{x3}\,l_{y2}}{l_{x1}\,l_y}+\frac{v^3l_{x4}\,l_{y3}}{l_{x1}\,l_y}+\ldots$ is equal to $a_{x1\cdot y}$) will become equal to $\frac{vl_{x1}\,l_y}{l_x\,l_y}(1+a_{x1\cdot y})=\frac{v(1+a_{x1\cdot y})l_{x1}}{l_x}$. In like manner, it will be evident that the fourth perpendicular series, or $\frac{vl_x\,l_{y1}}{l_x\,l_y}+\frac{v^2l_{x1}\,l_{y2}}{l_x\,l_y}+\frac{v^3l_{x2}\,l_{y3}}{l_x\,l_y}+\ldots$ is equal to $\frac{l_{1x}}{l_x}\left[\frac{vl_x\,l_{y1}}{l_{1x}\,l_y}+\frac{v^2l_{x1}\,l_{y2}}{l_{1x}\,l_y}+\frac{v^3l_{x2}\,l_{y3}}{l_{1x}\,l_y}+\ldots\right]$: which (since $\frac{vl_x\,l_{y1}}{l_{1x}\,l_y}+\frac{v^2l_{x1}\,l_{y2}}{l_{1x}\,l_y}+\frac{v^3l_{x2}\,l_{y3}}{l_{1x}\,l_y}+\ldots$ is equal to $a_{1x\cdot y}$) will become equal to $\frac{a_{1x\,y}\,l_{1x}}{l_x}$.

The value of these series, however, may be expressed in a different manner, by inverting the method here pursued: for, the third perpendicular series is (on the assumption just mentioned) also equal to $\frac{l_{1y}}{l_y}\left[\frac{vl_{x1}\,l_y}{l_x\,l_{1y}}+\frac{v^2l_{x2}\,l_{y1}}{l_x\,l_{1y}}+\frac{v^3l_{x3}\,l_{y2}}{l_x\,l_{1y}}+\ldots\right]=a_{x\cdot 1y}\times\frac{l_{1y}}{l_y}$: and the fourth perpendicular series is also equal to $\frac{vl_x\,l_{y1}}{l_x\,l_y}\left[1+\frac{vl_x\,l_{y1}}{l_x\,l_{y1}}+\frac{v^2l_{x2}\,l_{y3}}{l_x\,l_{y1}}+\frac{v^3l_{x3}\,l_{y4}}{l_x\,l_{y1}}+\ldots\right]=\frac{vl_x\,l_{y1}}{l_xl_y}(1+a_{x\cdot y1})=v(1+a_{x\cdot y1})\frac{l_{y1}}{l_y}$. Whence it appears that—

$$v(1+a_{x1\cdot y})\times\frac{l_{x1}}{l_x}=a_{x\cdot 1y}\times\frac{l_{1y}}{l_y}$$

$$v(1+a_{x\cdot y1})\times\frac{l_{y1}}{l_y}=a_{1x\cdot y}\times\frac{l_{1x}}{l_x}:$$

and consequently (since *each* series may be summed up in two different ways) that we may adopt either expression for the value of the same: a circumstance of which I have availed myself in some of the subsequent problems. The same observations will apply to any other two lives.

Mr. Morgan (apparently not aware of this property) has run into some strange errors on this subject. He has given *both* expressions in the *same* formula; and such expressions, being there always used with *contrary* signs, of course destroy each other, and are therefore *unnecessarily* introduced! These cases are by no means singular, as they occur in almost *all the problems* (involving similar contingencies) inserted by him in the *Philosophical Transactions*. See a remarkable instance of this kind in the note to the first case of Problem XLV.]

As this formula will often be referred to in the subsequent problems, it will be convenient to denote it by a more simple expression; therefore let it be represented by A_B; that is, let A_B denote the present value of £1 to be received on the contingency mentioned in the problem; consequently $s \times A_B$ will denote the present value of the given sum under the same circumstances.

Note.—The only methods of solving this problem (previous to the investigation of it by Mr. Morgan from the real probabilities of life) were those given by Mr. Simpson in the *Supplement* to his *Doctrine of Annuities*, Prob. XXXII., and by Mr. Dodson in the third volume of his *Mathematical Repository*, Ques. 23, both of which are deduced from M. De Moivre's hypothesis. Mr. Simpson, however, has inserted a rule which may be applied to any table of observations; whilst Mr. Dodson still abides by the hypothesis of his friend De Moivre. Mr. Morgan, with his usual antipathy against that theory, has despatched Mr. Dodson's rule in very few words, by saying that "having derived his rule from a wrong hypothesis, he has rendered it of *no use*" (*Phil. Trans.* vol. lxxviii. p. 332): and Dr. Price has observed that he knows of *no other* method of solution but that of Mr. Simpson. Had these gentlemen, however, taken the trouble to compare Mr. Simpson's rule with that given by Mr. Dodson, they would have found that the latter (even on De Moivre's hypothesis) gives the values oftentimes more correctly than the former; and were it adapted to the real probabilities of life, in the same manner as they have adapted Mr. Simpson's, it would in many cases give the values still more corresponding to the *true* values. I agree with them, however, that neither of these methods can be safely used, except in the middle stages of life; and that it will be best in *all* cases to deduce the values from real observations. Nevertheless, as a *near* value may be oftentimes required, without much labour of computation, I have thought it necessary to point out how far these methods may be depended on.

But, is it not singular that, after the unlimited censure which Mr. Morgan has cast upon the "wretched" hypothesis of De Moivre, he should (as editor of Dr. Price's *Observations on Reversionary Payments*) suffer the fortieth table in that work to remain without a comment? particularly, as he must well know, because he has taken much pains to prove, that the values in that table are *extremely erroneous*, and in many cases are more than *one-third* of their true value too much. And is it not more singular that, to this very hour, it should serve to determine the value of such assurances at the Equitable Society, the business of which Mr. Morgan has so long and so ably conducted! It is true that the concerns of that Society being, for the most part, established upon such fair and truly equitable principles, it little signifies *how much* is paid for an assurance provided every one pays a *proportion;* but it must be evident that, in the present case, the rest of the Society are benefited at the expense of those who assure on the contingency mentioned in this problem. And it is amusing to observe how blindly the other established Offices, as well as the new ephemeral Companies, have followed this error of their great prototype.

I have inserted the table, above alluded to, at the end of the present work (being the fourteenth table there given), not only as a matter of curiosity, but to show upon what erroneous principles the business of those Offices is sometimes conducted. See the Scholium to Question 30 in Chapter XII.]

COROLLARY I.

§ 232. Having thus found the present value of the given sum on the contingency of B surviving A, the present value of the same sum on the contingency of A surviving B (that is, of the same sum to be received at the death of B, provided his life be the first that becomes extinct) is readily found by substituting the symbols l_x, l_{x1}, l_{x2}, &c., for l_y, l_{y1}, l_{y2}, &c., in the above analysis: whence such value will come out equal to $\frac{s}{2}\left[v(1-ra_{xy})-\frac{v(1+a_{x\cdot y1})l_{y1}-a_{x\cdot y1}\,l_{1y}}{l_y}\right]$.

But, if the present value of the given sum to be received on the death of A according to the problem be once determined, we may easily find the value of the same sum to be received on the death of B, provided his life be the first that fails, by subtracting the value found by the problem from the present value of the given sum to be received on the extinction of the joint lives A B, as found by Prob. XXII.; or, in other words, by changing the sign of the second term in the general expression deduced from the problem. Thus $\frac{s}{2}\left[v(1-ra_{xy})+\frac{v(1+a_{x1\cdot y})\,l_{x1}-a_{1x\cdot y}\,l_{1x}}{l_x}\right]$ will also denote the present value of the given sum on the contingency of A surviving B.[1]

As these formulæ will be often referred to in the subsequent problems, it will be more convenient to denote them by a more simple expression; therefore let them be represented by B_A; that is, let B_A denote the present value of £1 to be received on the contingency mentioned in this corollary; consequently $s \times B_A$ will denote the present value of the given sum under the same circumstances.

COROLLARY II.

§ 233. From the principles laid down in the problem we may readily find the value of any sum depending on the contingency of C surviving A; that is, of such sum to be received on the death of A provided his life be the first that becomes extinct of the two lives A, C. The value of such sum will be equal to $\frac{s}{2}\left[v(1-ra_{xz}) - \frac{v(1+a_{x1\cdot z})\, l_{x1} - a_{1x\cdot z}\, l_{1x}}{l_x} \right]$; and which I shall (for reasons above mentioned) denote by $s \times A_C$.

In like manner the value of any sum depending on the contingency of C surviving B will be found equal to $\frac{s}{2}\left[v(1-ra_{yz}) - \frac{v(1+a_{y1\cdot z}) l_{y1} - a_{1y\cdot z} l_{1y}}{l_y} \right]$; which I shall, for the same reasons, denote by $s \times B_C$. These formulæ will often be referred to in the subsequent problems.

COROLLARY III.

§ 234. If the contingency mentioned in the problem, and on which the given sum is to be received, is *Deferred* for any given number of years $(=m)$, less than that to which it is probable the two joint lives may extend, it is evident that the several perpendicular series, in the general expression given in page 115, must be continued from after the mth term only; and thence to the utmost extent of human life. The sum of which terms will, by an attention to the steps pursued in the Problem XXII., cor. 1, be found equal to $\frac{s}{2}\Big[v(1 - ra_{xy\,|m} \times \frac{v^m l_x\, l_{y\,|m}}{l_x\, l_y} - v(1 + a_{x1\cdot y\,|m} \times \frac{v^m l_{x1}\, l_y}{l_x\, l_y} + a_{1x\cdot y} \times \frac{v^m l_{1x}\, l_y}{l_x\, l_y} \Big] = \frac{v^{m+1}\, s l_{y+m}}{2 l_x\, l_y} \times [(1 - ra_{x1y|m})\, l_{x+m} - (1 + a_{x1\cdot y|m})$

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[1] Universally we shall find that, by this method of investigation, the expression

$$\frac{s}{2}\left[v(-ra_{xy} \mp) \frac{v(1+a_{x1\cdot y})\, l_{x1} - a_{1x\cdot y}\, l_{1x}}{l_x} \right] =$$

$$\frac{s}{2}\left[v(1-ra_{xy}) \pm \frac{v(1+a_{x\cdot y1})\, l_{y1} - a_{x\cdot 1y}\, l_{1y}}{l_y} \right]$$

will denote the value in either case; where the upper sign is to be taken if the sum depends on the contingency of A's dying first; and the lower sign, if on B's dying first.

$\times l_{x1|m} + a_{1x\cdot y|m}(1+r)\,l_{1x+m}]$:[1] an expression which I shall denote by $s \times A_{B(m}$.

COROLLARY IV.

§ 235. Or, if the contingency, on which the given sum is to be received, is *Temporary* (that is, to continue only for a given *term*$=m$, less than that to which it is probable the two joint lives may extend), it is evident that the several perpendicular series, in the general expression given in page 115, must be continued for m terms only: and the sum of these will, by an attention to the steps pursued in Prob. XXII. cor. 2, be found equal to $\frac{s}{2}\left[v(1+a_{xy)m-1}) - a_{xy)m} - \frac{v(1+a_{x1\cdot y)m-1}\,l_{x1})}{l_x} - a_{1x\cdot y)m} \right]$.

But since $a_{x1\cdot y)m-1} + \frac{v^m l_{x1}\,l_{y\,|m}}{l_{x1}\,l_y} = a_{x1\cdot y)m} = a_{x1\cdot y} - a_{x1\cdot y|m} \times \frac{v^m l_{x1}\,l_{y|m}}{l_{x1}\,l_y}$, we may (by substituting these values, as explained in the problem just referred to) render this formula more convenient for practice by the following ing expression $\frac{s}{2}\left[v(1-ra_{xy}) - v(1-ra_{xy|m}) \times \frac{v^m l_x\,l_{y\,|m}}{l_x\,l_y} - \frac{v(1+a_{x1\cdot y})\,l_{x1}}{l_x} + \frac{v(1+a_{x1\cdot y|m})\,v^m l_{x1\cdot y|m}}{l_x\,l_y} + \frac{a_{1x\cdot y}\,l_{1x}}{l_x} - \frac{a_{1x\cdot y|m}\,v^m l_{1x\cdot y}}{l_x\,l_y} \right] = sA_B - sA_{B(m}$: that is, equal to the difference between the present value of the sum found by the problem, and the present value of the same sum *Deferred* as in the preceding corollary.

I believe that this formula is the first *correct* rule that has ever been given for determining the value of such assurances for given *terms*. In the solution of the fifteenth question in Dr. Price's *Obs. on Rev. Pay.*, the author has given a rule by which we may obtain a *near* value of such assurances, and has referred to Note (H) for a *demonstration* of the same. This demonstration (as he calls it) would, indeed, lead us to the *exact* value according to any table of observations agreeably to what the author has justly remarked: the *rule*, however, has not been deduced from this investigation, but from Question 23 in the third volume of Mr. Dodson's *Mathematical Repository*, or from the process laid down by Mr. Simpson in the *Supplement* to his *Doctrine of Annuities*, p. 62.

Mr. Morgan, by way of improving on this *explanation*, has given a rule (without a demonstration) which he says is obtained "without the aid of M. De Moivre's hypothesis." This, however, is not strictly true; for his rule is deduced on the supposition that the decrements of life are equal during the given term; similar to the method explained in the note in page 97 of the present work: and this use of M. De Moivre's hypothesis but little accords with what he has repeatedly advanced, "that it is best to exclude it *entirely* from the doctrine of survivorship." This very fact will show that it may occasionally be used with good effect, even in the solution of the very problem where he has censured it the most. I shall here subjoin an investigation of Mr. Morgan's rule, which will show the method of proceeding in similar cases, when we wish to adopt the same principles.

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[1] It must be remembered that $a_{x1\cdot y|m}$ denotes the value of an annuity on two joint lives, each m years older than $x+1$ and y; and that $a_{1x\cdot y|m}$ denotes the value of an annuity on two joint lives, each m years older than $x-1$ and y respectively: agreeably to what has been said in § 229.

It will appear that the general series in the problem may be converted into the following one:—

$$\frac{svd_x}{2l_x}(\frac{l_y+l_{y1}}{l_y})+\frac{sv^2d_{x1}}{2l_x}(\frac{l_{y1}+l_{y2}}{l_y})+\frac{sv^3d_{x2}}{2l_x}(\frac{l_{y2}+l_{y3}}{l_y})+\dots$$

Now, if we suppose the decrements of A's life to be equal during the given term (which in many cases will be found to correspond very nearly with the fact), and if such decrements be denoted by $\delta=\frac{l_x-l_{x+m}}{m}$, then will the first m terms of this series be equal to

$$\frac{s\delta v}{2l_x}(\frac{l_y}{l_y}+\frac{l_{y1}}{l_y})+\frac{s\delta v^2}{2l_x}(\frac{l_{y1}}{l_y}+\frac{l_{y2}}{l_y})+\frac{s\delta v^3}{2l_x}(\frac{l_{y2}}{l_y}+\frac{l_{y3}}{l_y})+\dots\frac{s\delta v^m}{2l_x}(\frac{l_{1y+m}}{l_y}+\frac{l_{y+m}}{l_y}).$$

But the first of these series, within the parentheses, is (independent of the common multiple $\frac{s\delta}{2l_x}$), equal to $v(1+a_{y)m-1})$; and the second is, in like manner, equal to $a_{y)m}$: whence the total present value of the assurance for the given term will, upon this hypothesis, be denoted by

$$\frac{s\delta}{2l_x}[v(1+a_{y)m-1})+a_{y)m}]=s\frac{(l_x-l_{x+m})}{2l_xm}v(1+a_{y)m-1}+a_{y)m}):$$

and whence Mr. Morgan's rule is manifest.

But this rule may be more conveniently expressed by $\frac{s(l_x-l_{x+m})}{2l_xm}\times v\left[1+(2+r)a_y-[1+(2+r)a_{y+m}]\times\frac{v^ml_{y+m}}{l_y}\right]$; which, when the two series are continued to the utmost extent of human life (B being the oldest), will become $\frac{s(l_x-l_{x+m})}{2l_xm}\times v[1+(2+r)a_y]$. In this latter case, however, the formula cannot be depended upon, unless A is much the youngest of the two lives, but, at the same time, not younger than 15 years of age; indeed, it must at *all* times be used with proper caution, and with a due regard to the tables of observations made use of. These remarks will equally apply to the solution of the sixteenth question in Dr. Price's work, above quoted.

COROLLARY V.

§ 236. If the two lives are equal to each other, or of the same age A, the general expression in the problem becomes equal to $\frac{sv}{2}(1-ra_{xx})$, because the second term in such expression, or $\frac{v(1+a_{x1\cdot x})\,l_{x1}-a_{1xx}\,l_{1x}}{l_x}$, will then vanish altogether: whence it follows that the value will in such case be equal to one half the present value of the given sum to be received on the extinction of the joint lives.

This observation will equally apply to the formulæ, in the last two corollaries, for *terms*, and is indeed self-evident. For, as the given sum is not to be received till after the extinction of the joint lives, and as it is certainly an *even* chance which of the two may survive, it is manifest that one *half* the present value of the sum to be received on the extinction of those joint lives will be the true value of the sum depending on this contingency; whether such contingency extends to the whole continuance of the given lives, or exists only for a term.

Observations on Mr. Morgan's Method of investigating this Problem.

§ 237. I cannot dismiss this problem without some remarks on the singular and unscientific manner in which Mr. Morgan has summed up the four collateral series (in page 115) which express the present value of the given sum to be received on the contingency mentioned in the problem; and I am the more induced to do this because similar expressions arise in many of his subsequent problems, in which the same loose and eccentric method has been pursued; whereby an air of obscurity and difficulty has been thrown over the subject, which it is my intention here to remove. Mr. Morgan divides the whole expression into the two following collateral series,[1] viz.:—

$$\begin{aligned}
&\frac{sv}{2}\left(\frac{d_x\,l_{y1}}{l_x\,l_y}+\frac{d_x\,l_y}{l_x\,l_y}\right)+\\
&\frac{sv^2}{2}\left(\frac{d_{x1}\,l_{y2}}{l_x\,l_y}+\frac{d_{x1}\,l_{y1}}{l_x\,l_y}\right)+\\
&sv^3\left(\frac{d_{x2}\,l_{y3}}{l_x\,l_y}+\frac{d_{x2}\,l_{y2}}{l_x\,l_y}\right)+\\
&\qquad\&c.\qquad\qquad\&c.
\end{aligned}$$

which is evidently the same series as that which I have given in page 115.

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[1] In the subsequent process given in the text, I have adopted the method of notation made use of throughout the whole of this work, in order to prevent any obscurity or circumlocution by the introduction of new symbols. But, lest I should be charged with having misrepresented Mr. Morgan, I shall here subjoin the method which he himself has pursued in investigating this problem; and for the sake of greater accuracy shall adopt the same characters which he has used.

Let r denote £1 increased by its interest for a year; let a represent the number of per-

§ 238. The first of these collateral series he expands into a more complex one, by making it equal to

$$\begin{aligned}&\frac{sv}{2}\left(\frac{l_{y1}}{l_y}-\frac{l_{x1}\,l_{y1}}{l_x\,l_y}\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\right)+\\ &sv^2\left(\frac{l_{y2}}{l_y}-\frac{l_{x2}\,l_{y2}}{l_x\,l_y}-\frac{l_{y2}}{l_y}+\frac{l_{x1}\,l_{y2}}{l_x\,l_y}\right)+\\ &sv^3\left(\frac{l_{y3}}{l_y}-\frac{l_{x3}\,l_{y3}}{l_x\,l_y}-\frac{l_{y3}}{l_y}+\frac{l_{x2}\,l_{y3}}{l_x\,l_y}\right)+\\ &\qquad\qquad\text{etc.}\qquad\qquad\text{etc.}\end{aligned}$$

and he then proceeds to sum up *these* several collateral series in the following manner: The first he makes (independent of the common multiple $\frac{s}{2}$) equal to a_y, the second equal to $-a_{xy}$, the third equal to $-va_{y1}\times\frac{l_{y1}}{l_y}$,

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sons living in the table at the age of A; and let a', a'', a''', &c., represent the decrements at the end of the first, second, third, &c., years from the age of A. Let b represent the number of persons living at the age of B; and let c, d, e, f, &c., represent the number of persons living at the end of the first, second, third, fourth, &c., years from the age of B.

Then (in the words of Mr. Morgan, in *Phil. Trans.* for 1788, p. 242) it will follow that the present value of S to be received on the death of A, should that happen in the lifetime of B, will be expressed by the series $\text{S}\times\left[\frac{(b+c)a'}{2abr}+\frac{(c+d)a''}{2abr^2}+\frac{(d+e)a'''}{2abr^3}+\frac{(e+f)a''''}{2abr^4}+\text{\&c.}\right]$ This series may be resolved into the two following:—

$$\frac{\text{S}}{2}\times\left[\frac{ca'}{abr}+\frac{da''}{abr^2}+\frac{ea'''}{abr^3}+\frac{fa''''}{abr^4}+\text{\&c.}\right]+\frac{\text{S}}{2}\times\left[\frac{ba'}{abr}+\frac{ca''}{abr^2}+\frac{da'''}{abr^3}+\frac{ea''''}{abr^4}+\text{\&c.}\right].$$

The first of these two may be again resolved into

$$\frac{\text{S}}{2}\left[\frac{c}{br}-\frac{ca-ca'}{abr}+\frac{d}{br^2}-\frac{da-da'-da''}{abr^2}+\frac{e}{br^3}-\frac{ea-ea'-ea''-ea'''}{abr^3}+\text{\&c.}\right]$$
$$-\frac{\text{S}c}{2br}\times\left[\frac{d}{cr}-\frac{da-da'}{acr}+\frac{e}{cr^2}-\frac{ea-ea'-ea''}{acr^2}+\text{\&c.}\right]$$

Let B denote the value of an annuity on the life of B, C the value of an annuity on a life one year older than B, AB and AC the values of annuities on the joint lives of A and B and of A and C, and these series will be $=\frac{\text{S}\times(\text{B}-\text{AB})}{2}-\frac{\text{S}\times c(\text{C}-\text{AC})}{2br}$. Again, the second series above mentioned, or $\frac{\text{S}}{2}\times\left[\frac{ba'}{abr}+\frac{ca''}{abr^2}+\frac{da'''}{abr^3}+\text{\&c.}\right]$, by pursuing the same steps may be found $=\frac{\beta\text{S}}{2b}(\text{K}-\text{AK})-\frac{\text{S}\times(\text{B}-\text{AB})}{2r}$ where β denotes the number of persons living at the age of a person one year younger than B, K the value of an annuity on that life, and AK the value of an annuity on the joint lives of A and K. The whole value of the survivorship is therefore

$$=\text{S}\times\left[\frac{(r-1)\times(\text{B}-\text{AB})}{2r}+\frac{\beta(\text{K}-\text{AK})}{2b}-\frac{c(\text{C}-\text{AC})}{2br}\right].$$

These are Mr. Morgan's own words: and if the several values above alluded to be expressed by the characters which I have adopted in the present work, they will be found to agree with the process given in the text. But $\text{B}-\frac{c.\text{C}}{br}=\frac{c}{br}$, and $\frac{\beta.\text{K}}{b}-\frac{\text{B}}{r}=\frac{1}{r}$; whence the absurdity of his formula. Mr. Morgan, however, has at length altered his *formula* for this particular case; although he still preserves the same *method of solution* both here and in all his subsequent problems. See Price's *Obs.* edit. 1803, Note (O).

and the fourth equal to $va_{x\cdot y1}\times\frac{l_{y1}}{l_y}$: so that the total value of these four collateral series (or of the *first* collateral series in § 237) is made equal to $a_y - va_{y1}\times\frac{l_{y1}}{l_y} + va_{x\cdot y1}\times\frac{l_{y1}}{l_y} - a_{xy}$.

And, by proceeding in a similar manner with respect to the *second* collateral series in § 237, the sum of it is found equal to $a_{1y}\frac{l_{1y}}{l_y} - va_y - a_{x\cdot 1y}\frac{l_{1y}}{l_y} + va_{xy}$. Consequently, the total present value of the given sum, to be received on the contingency mentioned in the problem, is equal to $\frac{s}{2}\left[a_{1y}\frac{l_{y1}}{l_y} - va_y + va_{xy} - a_{xy} + a_y - va_{y1}\frac{l_{y1}}{l_y} + va_{x\cdot y1}\frac{l_{y1}}{l_y} - a_{x\cdot 1y}\frac{l_{1y}}{l_y}\right]$.

§ 239. But surely it cannot be necessary to tell Mr. Morgan that $a_y - va_{y1}\times\frac{l_{y1}}{l_y}$ is equal to $\frac{vl_{y1}}{l_y}$,[1] and that $a_{1y} + \frac{l_{1y}}{l_y} - va_y$ is equal to v only! Whereby the complex and troublesome formula above given may be more simply expressed by $\frac{s}{2}\left[v(1 - ra_{xy}) + v(1 + a_{x\cdot y1})\frac{l_{y1}}{l_y} - a_{x\cdot 1y}\times\frac{l_{1y}}{l_y}\right]$, and which is one of the formulæ which I have given, in the note to § 232, for the value of the sum depending on this contingency. Indeed, it is difficult to conceive why he should ever have thought it necessary to employ those quantities which denote the value of an annuity on a *single* life; since it is clear, from the nature of the several series in page 122, that such values could not properly belong to the subject; and that no quantities are involved in the solution, but such as arise from *two joint* lives.

As this strange error runs through the whole of Mr. Morgan's papers in the *Philosophical Transactions*, where this and similar contingencies are involved, I have thought it right to take this public notice of it, not only with a view to obviate any difficulty that may arise from a comparison of his formulæ with those which I have here deduced, but to clear the subject from all diffuse and unconnected matter. And I shall have occasion to make a similar remark on his method of summing the several series where *three* lives are concerned: for which, see the *Observations* in § 251, &c., at the end of Problem XXIX., and § 272 at the end of Problem XXXV.

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[1] This is indeed evident on inspection of the first and third of the collateral series in § 238. For it is manifest that all the subsequent terms of those two series *after the first terms,* destroy each other; and consequently that the sum of those two collateral series will be equal to the first term only; that is (independent of the common multiple $\frac{s}{2}$) equal to $\frac{vl_{y1}}{l_y}$. In like manner we may prove the equality of the other two quantities alluded to in the text.

PROBLEM XXVIII.[1]

§ 240. To determine the present value of a given sum payable on the decease of A, provided that shall be the *second* which fails of two given lives, A, B.

SOLUTION.

It is evident, in this case, that the chance of receiving the sum at the end of the first year will depend only on one contingency: viz., that both the lives become extinct in that period, restrained, however, to the condition that A dies last, the probability of which in the first year is $\frac{d_x d_y}{2l_x l_y}$; and which, being multiplied by sv, or the present value of the given sum, certain to be received at the end of the year, will give $\frac{sv}{2}\left(\frac{l_x l_y}{l_x l_y}+\frac{l_{x1} l_{y1}}{l_x l_y}-\frac{l_{x1} l_y}{l_x l_y}-\frac{l_x l_{y1}}{l_x l_y}\right)$ for the expectation of receiving such sum at the end of the first year.

In the second and following years, however, the chance of receiving the sum will depend on the happening of either of two events: 1. that both the lives fail in the year, A having died last; 2. that only A dies in the year, and that B dies in either of the preceding years. The probability that the first event will happen in the second year is $\frac{d_{x1} d_{y1}}{2l_x l_y}$; and the probability of the second event happening in the same period is $\frac{d_{x1}}{l_x}(1-\frac{l_{y1}}{l_y})$. These two values, therefore, being added together and multiplied by sv^2, will give $\frac{sv^2}{2}\left(\frac{2l_{x1}}{l_x}-\frac{2l_{x2}}{l_x}-\frac{l_x l_{y|1}}{l_x l_y}+\frac{l_x l_{y|2}}{l_x l_y}+\frac{l_{x2} l_{y1}}{l_x l_y}-\frac{l_{x1} l_{y2}}{l_x l_y}\right)$ for the expectation of receiving the given sum at the end of the second year.

In like manner, it will be found that the probability of the first event happening in the third year is $\frac{d_{x2} d_{y2}}{2l_x l_y}$; and the probability that the second event will happen in the same period is $\frac{d_{x2}}{l_x}(1-\frac{l_{y2}}{l_y})$: therefore these two values, being added together and multiplied by sv^3, will give $\frac{sv^3}{2}\left(\frac{2l_{x2}}{l_x}-\frac{2l_{x3}}{l_x}-\frac{l_{x2} l_{y2}}{l_x l_y}+\frac{l_{x3} l_{y3}}{l_x l_y}+\frac{l_{x3} l_{y2}}{l_x l_y}-\frac{l_{x2} l_{y3}}{l_x l_y}\right)$ for the expectation of receiving the sum at the end of the third year: and so on for every subsequent year, to the utmost extent of human life; the sum of all which yearly values, or the series[1]

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[1] Price, Ques. 12, and Note (M). Dodson, vol. iii. Quest. 24. Simpson, Prob. 28; and *Sup.* Prob. 33; also Prob. 3 in p. 72. De Moivre, Prob. 17. Morgan, Prob. 17; and in *Phil. Trans.* for 1788, Prob. 3, page 347.

$$\begin{array}{l}\frac{sv}{2}\left(\frac{2l_x}{l_x}-\frac{2l_{x1}}{l_x}-\frac{l_x\,l_y}{l_x\,l_y}+\frac{l_{x1}\,l_{y1}}{l_x\,l_y}+\frac{l_{x1}l_y}{l_x\,l_y}-\frac{l_x\,l_{y1}}{l_x\,l_y}\right)+\\ \frac{sv^2}{2}\left(\frac{2l_{x1}}{l_x}-\frac{2l_{x2}}{l_x}-\frac{l_x\,l_{y|1}}{l_x\,l_y}+\frac{l_x\,l_{y|2}}{l_x\,l_y}+\frac{l_{x2}\,l_{y1}}{l_x\,l_y}-\frac{l_{x1}\,l_{y2}}{l_x\,l_y}\right)+\\ \frac{sv^3}{2}\left(\frac{2l_{x2}}{l_x}-\frac{2l_{x3}}{l_x}-\frac{l_x\,l_{y|2}}{l_x\,l_y}+\frac{l_x\,l_{y|3}}{l_x\,l_y}+\frac{l_{x3}\,l_{y2}}{l_x\,l_y}-\frac{l_{x2}\,l_{y3}}{l_x\,l_y}\right)+\\ \qquad\qquad\&c. \qquad\qquad \&c. \qquad\qquad \&c.\end{array}$$

will be the total present value of the sum s to be received on the above contingency.

§ 241. But the sum of the first two of these perpendicular series is, by Prob. XXII., equal to the present value of the given sum to be received on the decease of A; that is, equal to $sv(1-ra_x)$: and the remaining four of these perpendicular series are the same as those produced in the last problem, with a contrary sign; and the sum of which is there denoted by $\frac{s}{2}\left[v(1-ra_{xy})-\frac{v(1+a_{x1\cdot y})\,l_{x1}-a_{1x\cdot y}\,l_{1x}}{l_x}\right]$. Consequently the total value of these series will be expressed by $s\,[v(1-ra_x)—A_B]$.

COROLLARY I.

§ 242. Having thus found the present value of the given sum on the contingency of A dying after B; the present value of the same sum on the contingency of B dying after A (that is, of the same sum to be received on the death of B, provided his life is the second that fails) is readily determined by subtracting the value above found from the present value of the given sum to be received on the extinction of the longest of the two lives; which, by Prob. XXII. cor. 3, is expressed by $sv\,[1-r(a_x+a_y-a_{xy})]$. Consequently, the present value of the given sum, on the contingency of B dying after A, will be equal to

$$\frac{s}{2}\left[v[1-r(2a_y-a_{xy})]-\frac{[v(1+a_{x1\cdot y})\,l_{x1}-a_{1x\cdot y}\,l_{1x}}{l_x}\right]=s\,[v(1-ra_y)-B_A].$$

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[1] The value of the first year's expectation has been found in the preceding part of the investigation, equal to $\frac{sv}{2}\left(\frac{l_x\,l_y}{l_x\,l_y}+\frac{l_x\,l_{y|1}}{l_x\,l_y}-\frac{l_{x1}\,l_y}{l_x\,l_y}-\frac{l_x\,l_{y1}}{l_x\,l_y}\right)$; an expression which is evidently equal to $\frac{sv}{2}\left(\frac{2l_x}{l_x}-\frac{2l_{x1}}{l_x}-\frac{l_x\,l_y}{l_x\,l_y}+\frac{l_{x1}\,l_{y1}}{l_x\,l_y}+\frac{l_{x1}\,l_y}{l_x\,l_y}-\frac{l_x\,l_{y1}}{l_y\,l_y}\right)$. This latter value has been assumed in order to correspond with the terms of the subsequent expectations, and hereby render the several collateral series complete. A similar method is pursued in many of the following problems.

COROLLARY II.

§ 243. If the contingency on which the sum is to be received is *Deferred* for a given number of years $(=m)$, less than that to which it is probable the life A may extend, it is evident that the sum of all the terms of the first two perpendicular series, after the mth year, will, by Prob. XXII. cor. 1, be equal to $sv(1-ra_{x+m})\frac{v^m l_{x+m}}{l_x}$; and the sum of all the similar terms of the remaining four perpendicular series will be denoted by the same expression as that which has been deduced in the second corollary to the preceding problem, but with a contrary sign. Therefore, if the value found by that corollary be subtracted from the present value of the assurance of the given sum after the extinction of the life A, provided that happens after the given term, the difference will be the value required.

COROLLARY III.

§ 244. If the contingency on which this sum is to be received commences immediately, but continues only for a certain number of years $(=m)$, less than that to which it is probable the life A may extend: or, in other words, if we wish to determine the value of a *Temporary* assurance of such sum, it is evident that the several perpendicular series given in the problem must be continued only for m terms. Now, the sum of the first m terms of the first two of these perpendicular series will, by Prob. XXII. cor. 2, be found equal to $v(1-ra_x)-v(1-ra_{x+m})\frac{v^m l_{x+m}}{l_x}$: from which, if we subtract the first m terms[1] of the four remaining series, as found by cor. 4 in the preceding problem, the difference will be the value of the assurance for the given term.[2]

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[1] When m is *equal* to, or *greater* than, the number of years between the age of B and that age in the table of observations at which human life becomes extinct, we must subtract the *whole* value of the four remaining series as found by the last problem; that is, the correct value in both such cases will be denoted by s multiplied into

$$v\left[1-ra_x-(1-ra_{x+m})\frac{v^m l_{x+m}}{l_x}\right]-A_B.$$

[2] See the note to cor. 4, in the last problem; where the process there pursued will enable us to find a *near* value of an assurance for a given *term*, according to the conditions of the present problem. For, if we deduct the near value, there deduced, from the near value of an assurance of the same sum on the life of A for the same term, as found by the process laid down in the note to Prob. XXII. cor. 2, the difference, or

$$\frac{s\delta}{2l_x}\left[2+\frac{1-v^m}{r}-v(1+a_y)_{m-1}-a_{y)m}\right],$$ will be the value required.

This formula is the same in fact as that given by Mr. Morgan in Note (H) of Dr. Price's *Obs. on Rev. Pay.;* and is another instance of the utility of De Moivre's hypothesis; but

COROLLARY IV.

§ 245. If the two lives are equal, or of the same age A, the general expression in the problem becomes (agreeably to what has been already said in cor. 5 of the last problem) equal to $\frac{sv}{2}[1-r(2a_x-a_{xx})]$; that is, equal to one half the present value of the given sum to be received on the extinction of the longest of the two lives.

PROBLEM XXIX.[1]

§ 246. To determine the present value of a given sum payable on the decease of A, provided he be the *first* that fails of three given lives, A, B, C.

SOLUTION.

In order to receive the given sum at the end of any one year, it is necessary that one or other of four different events should happen: 1. that all the three lives fail in the year, A having died first; 2. that A and B fail in the year, A having died first, and that C lives to the end of it; 3. that A and C fail in the year, A having died first, and that B lives to the end of it; 4. that only A dies in the year, and that B and C both live to the end of it. The probabilities of the happening of these several events in the first year are respectively $\frac{d_x d_y d_z}{3l_x l_y l_z}$, $\frac{d_x d_y d_{z1}}{2l_x l_y l_z}$, $\frac{d_x d_y l_{z1}}{2l_x l_y l_z}$, and $\frac{d_x l_y l_z}{l_x l_y l_z}$; which being added together and multiplied by sv, will give

$$\frac{sv}{6}\times\left(\frac{2l_x l_y l_z}{l_x l_y l_z}-\frac{2l_x l_y l_{z1}}{l_x l_y l_z}-\frac{2l_{x1} l_y l_z}{l_x l_y l_z}+\frac{2l_x l_{y1} l_{z1}}{l_x l_y l_z}+\frac{l_x l_{y1} l_z}{l_x l_y l_z}-\frac{l_{x1} l_y l_{z1}}{l_x l_y l_z}+\frac{l_x l_y l_{z1}}{l_x l_y l_z}-\right.$$

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it may be rendered more convenient for practice by the substitutions adopted in the note to cor. 4 of the preceding problem; whereby it will become

$$\left[\frac{s(l_x-l_{x+m})}{2l_xm}[2\times\frac{1-v^m}{r}-v\left(1+(2+r)a_y]-[1+(2+r)a_{y+m}]\frac{v^m l_{y+m}}{l_y}\right)\right].$$

When m is equal to, or greater than, the number of years between the age of B and that age in the table of observations when human life becomes extinct, l_{y+m} is equal to 0; and consequently $\left[1+(2+r)a_{y+m}\times\frac{v^m l_{y+m}}{l_y}\right]$ vanishes: whereby the formula in both these cases would become $\frac{s(l_x-l_{x+m})}{2l_xm}\left[2\times\frac{1-v^m}{r}-v[1+(2+r)a_y\right]$. But in all *such* cases the value is best obtained by the rule given in the preceding note; since it is then equal to the difference between the value of an assurance on the life A for the given *term*, and the value of an assurance on the *whole* life A provided he be the first that fails.

[1] Dodson, vol. iii. Ques. 32. Morgan, Prob. 19; and in *Phil. Trans.* for 1791, Prob. 1, p. 248.

$\left.\frac{l_{x1}\,l_{y1}\,l_z}{l_x\,l_y\,l_z}\right)$ for the expectation of receiving the sum at the end of the first year.

In like manner, it will be found that the probabilities of the happening of these several events in the second year are respectively $\frac{d_x\,d_y\,d_{z|1}}{3l_x\,l_y\,l_z}$, $\frac{d_{x1}\,d_{y1}\,l_{z2}}{2l_x\,l_y\,l_z}$, $\frac{d_{x1}\,d_{z1}\,l_{y2}}{2l_x\,l_y\,l_z}$, and $\frac{d_{x1}\,l_{y2}\,l_{z2}}{l_x\,l_y\,l_z}$: which being added together, and multiplied by *sv*, will give the expectation of receiving the sum at the end of the second year. And, by a similar process, we may find the expectation of receiving the sum at the end of the third and every succeeding year, to the utmost extent of human life; the sum of all which yearly values, or the series,—

$$\frac{sv}{6}\left[\frac{2l_x\,l_y\,l_z}{l_x\,l_y\,l_z}-\frac{2l_x\,l_y\,l_{z|1}}{l_x\,l_y\,l_z}-\frac{2l_{x1}\,l_y\,l_z}{l_x\,l_y\,l_z}+\frac{2l_x\,l_{y1}\,l_{z1}}{l_x\,l_y\,l_z}+\frac{l_x\,l_{y1}\,l_z}{l_x\,l_y\,l_z}-\frac{l_{x1}\,l_y\,l_{z1}}{l_x\,l_y\,l_z}+\frac{l_x\,l_y\,l_{z1}}{l_x\,l_y\,l_z}-\frac{l_{x1}\,l_{y1}\,l_z}{l_x\,l_y\,l_z}\right]+$$

$$\frac{sv^2}{6}\left[\frac{2l_x\,l_y\,l_{z|1}}{l_x\,l_y\,l_z}-\frac{2l_x\,l_y\,l_{z|2}}{l_x\,l_y\,l_z}-\frac{2l_{x2}\,l_{y1}\,l_{z1}}{l_x\,l_y\,l_z}+\frac{2l_{x1}\,l_{y2}\,l_{z2}}{l_x\,l_y\,l_z}+\frac{l_{x1}\,l_{y2}\,l_{z1}}{l_x\,l_y\,l_z}-\frac{l_{x2}\,l_{y1}\,l_{z2}}{l_x\,l_y\,l_z}+\frac{l_{x1}\,l_{y1}\,l_{z2}}{l_x\,l_y\,l_z}-\frac{l_{x2}\,l_{y2}\,l_{z1}}{l_x\,l_y\,l_z}\right]+$$

$$\frac{sv^3}{6}\left[\frac{2l_x\,l_y\,l_{z|2}}{l_x\,l_y\,l_z}-\frac{2l_x\,l_y\,l_{z|3}}{l_x\,l_y\,l_z}-\frac{2l_{x3}\,l_{y2}\,l_{z2}}{l_x\,l_y\,l_z}+\frac{2l_{x2}\,l_{y3}\,l_{z3}}{l_x\,l_y\,l_z}+\frac{l_{x2}\,l_{y3}\,l_{z2}}{l_x\,l_y\,l_z}-\frac{l_{x3}\,l_{y2}\,l_{z3}}{l_x\,l_y\,l_z}+\frac{l_{x2}\,l_{y2}\,l_{z3}}{l_x\,l_y\,l_z}-\frac{l_{x3}\,l_{y3}\,l_{z2}}{l_x\,l_y\,l_z}\right]+$$

&c. &c. &c.

will be the total present value of the given sum to be received on the contingency above mentioned.

§ 247. But the sums of these eight perpendicular and collateral series (independent of the common multiple *s*) are respectively equal to $\frac{v(1+a_{xyz})}{3}-\frac{a_{xyz}}{3}-\frac{v(1+a_{x1\cdot y\cdot z})}{3}\cdot\frac{l_{x1}}{l_x}+a_{1x\cdot y\cdot z}\cdot\frac{l_{1x}}{3l_x}+\frac{v(1+a_{x\cdot y1\cdot z})}{6}\cdot\frac{l_{y1}}{l_y}-a_{x\cdot 1y\cdot z}\cdot\frac{l_{1y}}{6l_y}+\frac{v(1+a_{x\cdot y\cdot z1})}{6}\cdot\frac{l_{1z}}{6l_z}$.[1] Whence, the sum of the first and second is (as in Prob. XXII.) equal to $\frac{s}{3}\times v(1-ra_{xyz})$; the sum of the third and fourth is equal to $-\frac{s}{3l_x}[v(1+a_{x\cdot y1\cdot z})\,l_{x1}-a_{1x\cdot y\cdot z}\,l_{1x}]$; the sum of the fifth and sixth is equal to $\frac{s}{6l_y}[v(1+a_{x\cdot y1\cdot z})\,l_{y1}-a_{x\cdot 1y\cdot z}\,l_{1y}]$; and the sum of the seventh and eighth is equal to $\frac{s}{6l_z}[v(1+a_{x\cdot y\cdot z1})\,l_z-a_{x\cdot 1y\cdot z}\,l_{1z}]$. Consequently, the total present value of the given sum to be received on the above contingency, will be equal to s multiplied into $\frac{v(1-ra_{xyz})}{3}-\frac{1}{3l_x}[v(1+a_{x1\cdot y\cdot z})\,l_{x1}-a_{1x\,y\cdot z}\,l_{1z}]+\frac{1}{6l_y}[v(1+a_{x\cdot y1\cdot z})\,l_{y1}-a_{x\cdot 1y\cdot z}\,l_{1y}]+\frac{1}{6l_x}\times[v(1+a_{x\cdot y\cdot z1})\,l_{z1}-a_{x\cdot y\cdot 1z}\,l_{1z}]$.

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[1] The sum of the first and second of these collateral series is found by the process laid down in Prob. XXII.; and the remaining ones, in the following manner:—The sum of the third perpendicular series, or $\frac{vl_{x1}\,l_y\,l_z}{l_x\,l_y\,l_z}+\frac{v^2l_{x2}\,l_{y1}\,l_{z1}}{l_x\,l_y\,l_z}+\frac{v^3l_{x3}\,l_{y2}\,l_{z2}}{l_x\,l_y\,l_z}+\ldots$ (by taking all the terms as affirmative and leaving out the common multiple $\frac{2s}{6}$) is evidently equal to $\frac{vl_{x1}\,l_y\,l_z}{l_x\,l_y\,l_z}\left[1+\frac{vl_{x2}\,l_{y1}\,l_{z1}}{l_{x1}\,l_y\,l_z}+\frac{v^2l_{x3}\,l_{y2}\,l_{z2}}{l_{x1}\,l_y\,l_z}+\frac{v^3l_{x4}\,l_{y3}\,l_{z3}}{l_{x1}\,l_y\,l_z}+\ldots\right]$: which, since

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$\frac{vl_{x2}\,l_{y1}\,l_{z1}}{l_{x1}\,l_y\,l_z}+\frac{v^2l_{x3}\,l_{y2}\,l_{z2}}{l_{x1}\,l_y\,l_z}+\frac{v^3l_{x4}\,l_{y3}\,l_{z3}}{l_{x1}\,l_y\,l_z}+$ &c., is equal to $a_{x1\cdot y\cdot z}$, will become equal to $\frac{vl_{x1}\,l_y\,l_z}{l_x\,l_y\,l_z}(1+a_{x1\cdot y\cdot z})=\frac{v(1+a_{x1\cdot y\cdot z})\,l_{x1}}{l_x}$. In like manner, it is evident that, by leaving out the common multiple $\frac{2s}{6}$ the fourth perpendicular series, or $\frac{vl_x\,l_{y1}\,l_{z1}}{l_x\,l_y\,l_z}+\frac{v^2l_{x1}\,l_{y2}\,l_{z2}}{l_x\,l_y\,l_z}+\frac{v^3l_{x2}\,l_{y3}\,l_{z3}}{l_x\,l_y\,l_z}+$ &c., is equal to $\frac{l_{1x}}{l_x}\left(\frac{vl_x\,l_{y1}\,l_{z1}}{l_{1x}\,l_y\,l_z}+\frac{v^2l_{x1}\,l_{y2}\,l_{z2}}{l_{1x}\,l_y\,l_z}+\frac{v^3l_{x2}\,l_{y3}\,l_{z3}}{l_{1x}\,l_y\,l_z}+\text{ \&c.}\right)$: which, since $\frac{vl_x\,l_{y1}\,l_{z1}}{l_{1x}\,l_y\,l_z}+\frac{v^2l_{x1}\,l_{y2}\,l_{z2}}{l_{1x}\,l_y\,l_z}+\frac{v^3l_{x2}\,l_{y3}\,l_{z3}}{l_{1x}\,l_y\,l_z}+$ &c., is equal to $a_{1x\cdot y\cdot z}$, will become equal to $a_{1x\cdot y\cdot z}\,\frac{l_{1x}}{l_x}$. Therefore the sum of the two series, taking the former of them as negative, agreeably to the general expression in the text, will be $-\frac{v(1+a_{x1\cdot y\cdot z})\,l_{x1}}{l_x}+\frac{a_{xyz}\,l_{1x}}{l_x}$; and which being multiplied by their common multiple $\frac{2s}{6}$, will produce $-\frac{3}{3_x}[v(1+a_{x1\cdot y\cdot z})\,l_{x1}-a_{1x\cdot y\cdot z}\,l_{1x}]$ for the value of the same. The same method must be pursued in order to find the sum of the next two series, and also of the last two: but enough has been here said to enable the reader to perform the operations without stating the process at large.

I would however observe, that these several series may be expressed by *other* formulæ than those given in the text; for, the third perpendicular series, taking all the terms as affirmative, and leaving out the common multiple $\frac{2s}{6}$, is also equal to $\frac{l_{1y}\,l_{1z}}{l_y\,l_z}\times\left[\frac{vl_{x1}\,l_y\,l_z}{l_x\,l_{1y}\,l_{1z}}+\frac{v^2l_{x2}\,l_{y1}\,l_{z1}}{l_x\,l_{1y}\,l_{1z}}+\frac{v^3l_{x3}\,l_{y2}\,l_{z2}}{l_x\,l_{1y}\,l_{1z}}+\text{\&c.}\right]$; which (since the sum of the terms within the brackets is equal to $a_{x\cdot 1y\cdot 1z}$) will become equal to $a_{x\cdot 1y\cdot 1z}\times\frac{l_{1y}\,l_{1z}}{l_y\,l_z}$: and

As this formula will often be referred to in the subsequent problems, it will be convenient to denote it by a more simple expression; therefore let it be represented by A_{BC}; that is, let A_{BC} denote the present value of £1, to be received on the above contingency: consequently $s \times A_{BC}$ will denote the present value of the given sum under the same circumstances.

COROLLARY I.

§ 248. If it were required to find the present value of the given sum payable on the decease of B, provided he be the first that fails of the three lives, we may readily obtain such value by substituting A for B, and B for A, in the investigation of the problem. Whence, the present value required would come out equal to s multiplied into $\frac{v(1-ra_{xyz})}{3}+$

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the fourth perpendicular series, leaving out the common multiple $\frac{2s}{6}$, is also equal to $\frac{vl_x\,l_{y1}\,l_{z1}}{l_x\,l_y\,l_z}+\left[1+\frac{vl_{x1}\,l_{y2}\,l_{z2}}{l_x\,l_{y1}\,l_{z1}}+\frac{v^2l_{x2}\,l_{y3}\,l_{z3}}{l_x\,l_{y1}\,l_{z1}}+\frac{v^3l_{x3}\,l_{y4}\,l_{z4}}{l_x\,l_{y1}\,l_{z1}}+\ldots\right]$; which, (since the sum of the terms within the brackets is equal to $1+a_{x\cdot y1\cdot z1}$) becomes equal to $\frac{v(1+a_{x\cdot y1\cdot z1})\,l_{y1}\,l_{z1}}{l_y\,l_z}$. Consequently, the sum of the third and fourth perpendicular series given in the text will also be denoted by $+\frac{s}{3l_y\,l_z}[v(1+a_{x\cdot y1\cdot z1})\,l_{y1}\,l_{z1}-a_{x\cdot 1y\cdot 1z}\,l_{1y}\,l_{1z}]$; and may be substituted at pleasure for the same. And universally we shall find that

$$\frac{v(1+a_{x1\cdot y\cdot z})l_{x1}}{l_x}=\frac{a_{x\cdot 1y\cdot 1z}\,l_{1y}\,l_{1z}}{l_x\,l_y}$$

$$\frac{a_{1x\cdot y\cdot z}\,l_{1x}}{l_x}=\frac{v(1+a_{x\cdot y1\cdot z1})l_{y1}\,l_{z1}}{l_y\,l_z}$$

$$\frac{v(1+a_{x\cdot y1\cdot z})l_{y1}}{l_y}=\frac{a_{1x\cdot y\cdot 1z}\,l_{1x}\,l_{1z}}{l_x\,l_z}$$

$$\frac{a_{x\cdot 1y\cdot z}\,l_{1y}}{l_y}=\frac{v(1+a_{x1\cdot y\cdot z1})l_{x1}\,l_{y1}}{l_x\,l_z}$$

$$\frac{v(1+a_{x\cdot y\,z1})\,l_{z1}}{l_z}=\frac{a_{1x\cdot 1y\cdot z}\,l_{1x}\,l_{1y}}{l_x\,l_y}$$

$$\frac{a_{x\cdot y\cdot 1z}\,l_{1z}}{l_z}=\frac{v(1+a_{x1\cdot y1\cdot z})l_{x1}\,l_{y1}}{l_x\,l_y}$$

It therefore appears that *each* series may be summed up in *two* different ways, and that we may adopt either mode of expression for the value of the same. For the sake of uniformity, I have, in this and the following problems, kept to those which are given in the text; but Mr. Morgan (by merely changing these expressions, one for the other, according to the seniority of the lives, and then treating them as *different* quantities) has thrown an air of obscurity and confusion throughout the whole of his investigations; as I shall point out in the *Observations* at the end of this problem. See § 251.

$\frac{1}{6l_x}[v(1+a_{x1\cdot y\cdot z})l_{x1}-a_{1x\cdot y\cdot z}\,l_{1x}]-\frac{1}{3l_y}[v\,(1+a_{x\cdot y1\cdot z})\,l_{y1}-a_{x\cdot 1y\cdot z}\,l_{1y}]+\frac{1}{6l_z}$ $[v(1+a_{x\cdot y\cdot z1})l_{z1}-a_{x\cdot y\cdot 1z}\,l_{1z}]=B_{AC}$. By a similar process it will be found that the present value of the given sum payable on the decease of C, provided he be the first that fails of the three lives, is equal to s multiplied into $\frac{v(1-ra_{xyz})}{3}+\frac{1}{6l_x}[v(1+a_{x1\cdot y\cdot z})l_{x1}-a_{1x\cdot y\cdot z}\,l_{1x}]+\frac{1}{6l_y}[v(1+a_{x\cdot y1\cdot z})l_{y1}-$ $a_{x\cdot 1y\cdot z}\,l_{1y}]-\frac{1}{3l_z}[v(1+a_{x\cdot y\cdot z1})l_{z1}-a_{x\cdot y\cdot 1z}\,l_{1z}]=C_{AB}$.[1] As these formulæ may occasionally be found of use, I have thought proper to insert them here.

COROLLARY II.

§ 249. If the three lives are equal, or of the same age A, the last three terms in each of the above expressions destroy each other; and the formula is then reduced to $\frac{s}{3}v(1-ra_{xxx})$: an expression which denotes one-third of the present value of the given sum to be received on the extinction of the three joint lives.

COROLLARY III.[2]

§ 250. If the contingency, on which the sum is to be received, continues only for a given *term* $(=n)$, the present value of such sum will be equal to the sum of the first n terms of the several series given in the problem: the method of determining which will be manifest from the many examples which have preceded.[3]

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[1] I have represented these complex formulæ by the more simple quantities B_A and C_{AB}, for the sake of a more convenient reference; and the following process will show with how little trouble they may be converted into each other. Let us make

$$\frac{1}{l_x}[v(1+a_{x1\cdot y\cdot z})l_{x1}-a_{1x\cdot y\cdot z}\,l_{1x}]=\mathrm{a},$$

$$\frac{1}{l_y}[v(1+a_{x\cdot y1\cdot z})l_{y1}-a_{x\cdot 1y\cdot z}\,l_{1y}]=\mathrm{b},$$

$$\text{and } \frac{1}{l_z}[v(1+a_{x\cdot y\cdot z1})l_{z1}-a_{x\cdot y\cdot 1z}\,l_{1z}]=\mathrm{c};$$

$$\text{then will } A_{BC}=\frac{v(1-ra_{xyz})}{3}-\frac{\mathrm{a}}{3}+\frac{\mathrm{b}}{6}+\frac{\mathrm{c}}{6}$$

$$B_{AC}=\frac{v(1-ra_{xyz})}{3}+\frac{\mathrm{a}}{6}-\frac{\mathrm{b}}{3}+\frac{\mathrm{c}}{6}$$

$$C_{AB}=\frac{v(1-ra_{xyz})}{3}+\frac{\mathrm{a}}{6}+\frac{\mathrm{b}}{6}-\frac{\mathrm{c}}{3}.$$

[2] Morgan, Prob. 31.

[3] We may in general obtain a near value of the sum, in this case, by the help of M. De

Observations on Mr. Morgan's Method of investigating this Problem.

§ 251. The motives which induced me to notice (in page 122) the strange method which Mr. Morgan has adopted in summing up the several series arising from the investigation of the 27th problem, must be my apology here for again detaining the reader, whilst I expose the equally diffuse and obscure manner which he has also adopted in summing the several series arising from the investigation of this problem.

It will be seen, from an inspection of the series in page 129, that such series may be expressed in the following manner :—

$$\begin{array}{l}
sv\left[\frac{d_x\,l_y\,l_z}{3l_x l_y\,l_z}+\frac{d_x\,l_{y1}\,l_z}{6l_x\,l_y\,l_z}+\frac{d_x\,l_y\,l_{z1}}{6l_x\,l_y\,l_z}+\frac{d_x\,l_{y1}\,l_{z1}}{3l_x\,l_y\,l_z}\right]+\\
sv^2\left[\frac{d_{x1}\,l_{y1}\,l_{z1}}{3l_x\,l_y\,l_z}+\frac{d_{x1}\,l_{y2}\,l_{z1}}{6l_x\,l_y\,l_z}+\frac{d_{x1}\,l_{y1}\,l_{z2}}{6l_x\,l_y\,l_z}+\frac{d_{x1}\,l_{y2}\,l_{z2}}{3l_x\,l_y\,l_z}\right]+\\
sv^3\left[\frac{d_{x2}\,l_{y2}\,l_{z2}}{3l_x\,l_y\,l_z}+\frac{d_{x2}\,l_{y3}\,l_{z2}}{6l_x\,l_y\,l_z}+\frac{d_{x2}\,l_{y2}\,l_{z3}}{6l_x\,l_y\,l_z}+\frac{d_{x2}\,l_{y3}\,l_{z3}}{3l_x\,l_y\,l_z}\right]+\\
\qquad\qquad\&c. \qquad\qquad \&c. \qquad\qquad \&c.
\end{array}$$

and this is the way in which Mr. Morgan has thought proper to represent the present value of the given sum, to be received on the contingency mentioned in the problem.[1]

Moivre's hypothesis, which will save much time and trouble. For the series given in the problem is evidently equal to

$$\begin{array}{l}
\frac{sv}{6}\times\frac{d_x}{l_x}\left[\frac{2l_y\,l_z}{l_y\,l_z}+\frac{2l_{y1}\,l_{z1}}{l_y\,l_z}+\frac{l_{y1}\,l_z}{l_y\,l_z}+\frac{l_y\,l_{z1}}{l_y\,l_z}\right]+\\
\frac{sv^2}{6}\times\frac{d_{x1}}{l_x}\left[\frac{2l_{y1}\,l_{z1}}{l_y\,l_z}+\frac{2l_{y2}\,l_{z2}}{l_y\,l_z}+\frac{l_{y2}\,l_{z1}}{l_y\,l_z}+\frac{l_{y1}\,l_{z2}}{l_y\,l_z}\right]+\\
\frac{sv^3}{6}\times\frac{d_{x2}}{l_x}\left[\frac{2l_{y2}\,l_{z2}}{l_y\,l_z}+\frac{2l_{y3}\,l_{z3}}{l_y\,l_z}+\frac{l_{y3}\,l_{z2}}{l_y\,l_z}+\frac{l_{y2}\,l_{z3}}{l_y\,l_z}\right]+\\
\qquad\qquad\&c. \qquad\qquad \&c. \qquad\qquad \&c.
\end{array}$$

which, being continued to n terms, will be the required value of the given sum. But, since the quantities d_x, d_{x1}, d_{x2}, &c., are supposed to be equal and uniform (that is, equal to $\delta=\frac{l_x-l_{x+m}}{n}$), the sum of the first n terms of these several perpendicular series will be equal to $s.\frac{l_x-l_{x+m}}{nl_x}$ multiplied into $\frac{v(1+a_{yz)m-1})}{3}+\frac{a_{yz)m}}{3}+\frac{v(1+a_{y1\cdot z)m-1})}{6}.\frac{l_{y1}}{l_y}+\frac{a_{1y\cdot z)m}}{6}.\frac{l_{1y}}{l_y}$. If A is the youngest of the three lives, and differs much from the ages of both the other lives, this formula will be tolerably correct, even if the several perpendicular series are continued to the utmost extent of human life ; and it will consequently serve in such cases to find a *near* value of the sum for the whole continuance of the lives : but it must always be used with proper caution.

[1] In this investigation I have adopted the same symbols which have been used throughout the whole of this work ; but, lest I may be accused of having misrepresented Mr. Morgan, I shall here give his own solution, and in his own style.

Let r denote £1 increased by its interest for a year. Let a denote the number of

§ 252. In order to sum up these four collateral series he proceeds in the following manner:—The *first* of them he expands (as in *Phil. Trans.* for 1789, p. 44) into the more complex series

$$sv\left[\frac{l_y\,l_z}{3l_y\,l_z}-\frac{l_{x1}\,l_y\,l_z}{3l_x\,l_y\,l_z}\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\right]+$$
$$sv^2\left[\frac{l_{y1}\,l_{z1}}{3l_y\,l_z}-\frac{l_{x2}\,l_{y1}\,l_{z1}}{3l_x\,l_y\,l_z}-\frac{l_{y1}\,l_{z1}}{3l_y\,l_z}+\frac{l_x\,l_y\,l_{z|1}}{3l_x\,l_y\,l_z}\right]\cdot+$$
$$sv^3\left[\frac{l_{y2}\,l_{z2}}{3l_y\,l_z}-\frac{l_{x3}\,l_{y2}\,l_{z2}}{3l_x\,l_y\,l_z}-\frac{l_{y2}\,l_{z2}}{3l_y\,l_z}-\frac{l_x\,l_y\,l_{z|2}}{3l_x\,l_y\,l_z}\right]+$$

the sum of all which (independent of the common multiple $\frac{s}{3}$) he makes equal to $a_{1y\cdot1z}\frac{l_{1y}\,l_{1z}}{l_y\,l_z}-a_{x\cdot1y\cdot1z}\times\frac{l_{1y}\,l_{1z}}{l_y\,l_z}-va_{yz}+va_{xyz}$.

By proceeding in a similar manner, he makes the *second* collateral series, in page 133 (independent of the common multiple $\frac{s}{6}$) equal to $a_{y\cdot1z}\frac{l_{1z}}{l_z}-a_{x\cdot y\cdot1z}\times\frac{l_{1z}}{l_z}-va_{y1\cdot z}\times\frac{l_{y1}}{l_y}+va_{x\cdot y1\cdot z}\times\frac{l_{y1}}{l_y}$; the *third* equal to $a_{1y\cdot z}\times\frac{l_{1y}}{l_y}-a_{x\cdot1y\cdot z}\times\frac{l_{1y}}{l_y}-va_{y\cdot z1}\times\frac{l_{z1}}{l_z}+va_{x\cdot y\cdot z1}\times\frac{l_{z1}}{l_z}$: and the *fourth* equal to $a_{yz}-a_{xyz}-va_{y1\cdot z1}\times\frac{l_{y1}\,l_{z1}}{l_y\,l_z}+va_{x\cdot y1\cdot z1}\times\frac{l_{y1}\,l_{z1}}{l_y\,l_z}$. Consequently, the sum of the four collateral series in page 133 (or the total present value of the given sum to be received on the contingency mentioned in the problem)

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persons living at the age of A; and a', a'', a''', &c., the decrements of life at the end of the first, second, third, &c., years from the age of A. Let b denote the number of persons living at the age of B; and m, n, o, p, &c., the number living at the end of one, two, three, &c., years from that age. Let c denote the number of persons living at the age of C; and d, e, f, &c., the number living at the end of one, two, three, &c., years from that age. Let F denote the value of an annuity on a life one year younger than B; and β the number of persons living at that age. Let K denote the value of an annuity on a life one year younger than C; and κ the number of persons living at that age. Let P denote the value of an annuity one year older than B; and T the value of an annuity one year older than C. And let the given sum be denoted by S.

Then (in the words of Mr. Morgan, *Phil. Trans.* for 1791, p. 249) the whole value of the given sum will be $=\frac{S}{3abc}\times\left[\frac{a'bc}{r}+\frac{a''md}{r^2}+\frac{a'''en}{r^3}+\text{\&c.}\right]+\frac{S}{6abc}\times\left[\frac{a'mc}{r}+\frac{a''nd}{r^2}+\frac{a'''oe}{r^3}+\text{\&c.}\right]+\frac{S}{6abc}\times\left[\frac{a'db}{r}+\frac{a''em}{r^2}+\frac{a'''fn}{r^3}+\text{\&c.}\right]+\frac{S}{3abc}\times\left[\frac{a'md}{r}+\frac{a''en}{r^4}+\frac{a'''fo}{r^3}+\text{\&c.}\right]$. But the first of these series (he observes) is $=\frac{S}{3}\times\frac{\beta\kappa(\text{FK}-\text{AFK})}{bc}-\frac{S}{3r}\times(\text{BC}-\text{ABC})$; the second is $=\frac{S}{6}\times\frac{\kappa(\text{EK}-\text{ABK})}{c}-\frac{S}{6r}\times\frac{m(\text{PC}-\text{APC})}{b}$; the third is $=\frac{S}{6}\times\frac{\beta(\text{FC}-\text{AFC})}{b}-\frac{S}{6r}\times\frac{d(\text{BT}-\text{ABT})}{c}$; the fourth is $=\frac{S}{3}\times(\text{BC}-\text{ABC})-\frac{S}{3r}\times\frac{md(\text{PT}-\text{APT})}{bc}$. Therefore these several expressions being added together will be found $=$ S into $\frac{\kappa}{3c}\times\left[\frac{\beta(\text{FK}-\text{AFK})}{b}\right.$

will, as he observes, be found equal to *s* multiplied into $a_{1y\cdot 1z}\times\frac{l_{1y}\,l_{1z}}{3l_y\,l_z}-a_{x\cdot 1y\cdot 1z}\times\frac{l_{1y}\,l_{1z}}{3l_y\,l_z}-\frac{va_{yz}}{3}+\frac{va_{xyz}}{3}+a_{y\cdot 1z}\times\frac{l_{1z}}{6l_z}-a_{x\cdot y\cdot 1z}\times\frac{l_{1z}}{6l_z}-va_{1y\cdot z}\times\frac{l_{y1}}{6l_y}+va_{x\cdot y1\cdot z}\times\frac{l_{y1}}{6l_y}+a_{1y\cdot z}\times\frac{l_{1y}}{6l_y}-a_{x\cdot 1y\cdot z}\times\frac{l_{1y}}{6l_y}-va_{y\cdot z1}\times\frac{l_{z1}}{6l_z}+va_{x\cdot y\cdot z1}\times\frac{l_{z1}}{6l_z}+\frac{a_{yz}}{3}-\frac{a_{xyz}}{3}-va_{y1\cdot z1}\times\frac{l_{y1}\,l_{z1}}{3l_y\,l_z}+va_{x\cdot y1\cdot z1}\times\frac{l_{y1}\,l_{z1}}{3l_y\,l_z}$.

§ 253. Now, a moment's attention to the steps of this process will convince us that all those quantities, in which *two* joint lives are involved, are *unnecessarily* introduced into the above formula; since they evidently destroy each other, and may consequently be made to vanish altogether. For,

$$a_{1y\cdot 1z}\times\frac{l_{1y}\,l_{1z}}{l_y\,l_z}-va_{yz}=v,^{1}$$

$$a_{y\cdot 1z}\times\frac{l_{1z}}{l_z}-va_{1y\cdot z}\times\frac{l_{y1}}{l_y}=\frac{vl_{y1}}{l_y}$$

$$a_{1y\cdot z}\times\frac{l_{1y}}{l_y}-va_{y\cdot z1}\times\frac{l_{z1}}{l_z}=\frac{vl_{z1}}{l_z}$$

$$a_{yz}-va_{y1\cdot z}\times\frac{l_{y1}\,l_{z1}}{l_y\,l_z}=\frac{vl_{y1}\,l_{z1}}{l_y\,l_z}:$$

$+\frac{\text{BK}-\text{ABK}}{2}\Big]+\frac{\beta}{6b}\times(\text{FC}-\text{AFC})+\frac{r-1}{3r}\times(\text{BC}-\text{ABC})-\frac{m(\text{PC}-\text{APC})}{6br}-\frac{d}{3cr}\times\left[\frac{\text{BT}-\text{ABT}}{2}+\frac{m(\text{PT}-\text{APT})}{b}\right]$.

Thus far Mr. Morgan.—And these several values, being expressed by the characters made use of in the present work, will be found to agree with the process given in the text. But it may be demonstrated that

$$\frac{\beta\kappa.\text{FK}}{bc}-\frac{\text{BC}}{r}=\frac{1}{r}$$

$$\frac{\kappa.\text{BK}}{c}-\frac{m.\text{PC}}{br}=\frac{m}{br}$$

$$\frac{\beta.\text{FC}}{b}-\frac{d.\text{BT}}{cr}=\frac{d}{cr}$$

$$\text{BC}\;-\frac{dm.\text{PT}}{bcr}=\frac{dm}{bcr}:$$

whence the complex formula above given may be divested of all those expressions involving two joint lives, and may be more simply expressed by S multiplied into $\frac{1-(r-1)\text{ABC}}{3r}+\frac{1}{3bc}\times\left[\frac{(1+\text{APT})dm}{r}-\text{AFK}.\beta\kappa\right]\times\frac{1}{6b}\left[\frac{(1+\text{APC})m}{r}-\text{AFC}.\beta\right]+\frac{1}{6c}\left[\frac{1(+\text{ABT})d}{r}-\text{ABK}\kappa\right]$: which corresponds with the formula given in the text.

[1] It is evident, *on inspection*, that this is the value of the first and third collateral series in page 134; because all the subsequent terms *after the first term* manifestly destroy each other. Consequently the sum of those two collateral series will be equal to the first

whence, the complex formula given by Mr. Morgan may be reduced to s multiplied into $\frac{v(1-ra_{xyz})}{3}+\frac{1}{3l_y\,l_z}[v(1+a_{x\cdot y1\cdot z1})l_{y1}\,l_{z1}-a_{x\cdot 1y\cdot 1z}\times l_{1y}\,l_{1z}]+\frac{1}{6l_y}[v(1+a_{x\cdot y1\cdot z})\,l_y-a_{x\cdot 1y\cdot z}\,l_{1y}]+\frac{1}{6l_z}[v(1+a_{x\cdot y\cdot z1})\,l_{z1}-a_{x\,y\cdot 1z}\,l_{1z}]$: and which is the very same as that which I have given in page 130, except as to the substitution of the second term $+\frac{1}{3l_y\,l_z}[v(1+a_{x\cdot y1\cdot z1})l_{y1}\,l_{z1}-a_{x\cdot 1y\cdot 1z}\,l_{1y}\,l_{1z}]$ for its equal $-\frac{1}{3l_x}[v(1+a_{x1\cdot y\cdot z}\,l_x-a_{1x\cdot y\cdot z}\,l_{1x}]$, agreeably to what I have said in the note in page 131.

From this it will be readily seen that Mr. Morgan's formula consists of *double the number of terms* that are requisite for solving the problem; whereby it is rendered not only unnecessarily intricate and laborious, but also liable to considerable error in the *arithmetical* solution. Indeed it is manifest, from the nature of the series in page 129, that no expression involving *two* joint lives can properly arise: and it is singular that this consideration did not induce Mr. Morgan (if his object were perspicuity and information) to retrace the steps of his investigation, in order to render the process at once more simple and clear.

§ 254. But there is another part of Mr. Morgan's investigation on which I think it equally necessary to make some observations. He states that the theorem which he has deduced, as above, "gives the exact value

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term only; that is (independent of the common multiple $\frac{s}{3}$), equal to $v\frac{l_y\,l_z}{l_y\,l_z}=v$. In like manner may be demonstrated the truth of the remaining equation.

I shall, however, make another observation here, which I think of considerable importance. Although these equations are strictly and mathematically correct, yet (owing to the imperfection of the Tables which show the value of Life Annuities) they oftentimes vary in a material degree when solved *arithmetically:* an inaccuracy which, being repeated so frequently in the *same* formula, must render the solution, according to Mr. Morgan's investigation, extremely erroneous.

Thus, the compound quantity $a_{1y\cdot 1z}\frac{l_{1y}\,l_{1z}}{l_y\,l_z}-va_{yz}$ is equal to v, and (if simplicity and elegance were out of the question, and no regard were paid to the labour and intricacy of the solution) might be substituted instead. Let us, however, reduce these expressions to *numbers;* for which purpose let us take the age of B 15 years, the age of C 18 years, the rate of interest 4 per cent., and the probabilities of living as at Northampton. In this case we should have $v=\cdot962$; but the value of $a_{1y\cdot 1z}\frac{l_{1y}\,l_{1z}}{l_y\,l_z}-va_{yz}$ is equal to 13·566—12·614=·952 only; a difference which arises from the want of more enlarged tables of the value of annuities on two joint lives, and which will have a considerable effect upon the proper solution of the problem. This remark extends to *all* the formulæ inserted by Mr. Morgan in the several volumes of the *Philosophical Transactions;* and will show that those formulæ, however correct in principle, are of little or no use in *practice.*

when either B or C is the oldest of the three lives; but when A is the oldest, it will be *necessary*[1] to change the symbols, &c.:". and he has deduced *another* formula, for the solution of the problem in this case, involving *other* characters and symbols which render the investigation still more confused. From all which he would seem to imply that the first formula was not applicable to the second case.

But the second formula which he has deduced is still the *same* (mixed however with the useless and extraneous quantities above alluded to) as that which I have given in page 130; except as to the exchange of the last term $+\frac{1}{6l_z}[v(1+a_{x\cdot y\cdot z1})l_{z1}-a_{x\cdot y\cdot 1z}\,l_{1z}]$, for its equal $-\frac{1}{6l_x\,l_y}[v(1+a_{x1\cdot y1\cdot z})$ $\times\, l_{x1}\,l_{y1}-a_{1x\cdot 1y\cdot z}\,l_{1x}\,l_{1y}]$, agreeably to what I have said in the note in page 131.[2] And it may be here useful to remark that the formula, which I have given in page 131, is *universally* true, and does not depend on the seniority of the lives concerned. The substitutions above mentioned are mere arbitrary arrangements for the numerical solution of the problem; which may be occasionally used, or not, at the option of the calculator.

§ 255. It is easy to perceive, from what is here said, that the formula (denoting the value of the sum depending on the contingency in this problem) may be varied in *three* different ways, merely by the manner of summing up the several collateral series given in page 129; agreeably to the method laid down in the note in page 130. But such variation is by no means necessary or useful. Mr. Morgan has, in a very circuitous and diffuse manner, demonstrated only *two* of these ways:[3] but, by this additional attempt at elucidation, he has certainly rendered the subject still more confused. The introduction, indeed, of *unnecessary quantities* into any investigation (but more particularly the retaining of them in any resulting formulæ), and the capricious changing of the symbols employed, ought to be universally reprobated; not only as subversive of the true ends of science (whose object is information and not mystery), but also as destructive of all good taste in mathematical reasoning.

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[1] It certainly is not *necessary* to change the symbols; since, in whatever manner the characters are varied, we must come to the same general series at last: neither does such a measure facilitate the means of investigation. Indeed, I am at a loss to conceive why Mr. Morgan should have attempted to render so obscure this interesting branch of the subject. As a display of ingenuity it must completely fail; since his papers will be read with very different emotions from those of admiration.

[2] Mr. Morgan's formula, expressed by his own ambiguous characters, is in this case equal to S multiplied into $\frac{\beta}{3b}\times\left[\frac{a(\mathrm{HF}+\frac{1}{2}\mathrm{HFC})}{a}-(\mathrm{AF}+\frac{1}{2}\mathrm{AFC})\right]+\frac{1}{6}\times\left[\frac{a(\mathrm{HB}+2\mathrm{HBC})}{a}-\right.$ $\left.(\mathrm{AB}+2\mathrm{ABC})\right]+\frac{1}{3r}\times\left[\frac{s(\mathrm{BN}-\mathrm{BNC})}{a}-(\mathrm{AB}-\mathrm{ABC})\right]+\frac{m}{6br}\times\left[\frac{s(\mathrm{PN}-\mathrm{PNC})}{a}-\right.$ $\left.(\mathrm{AP}-\mathrm{APC})\right]$. Where H and N denote the value of an annuity on a life one year younger and one year older than A respectively; and where a and s denote the number of persons living at those ages respectively; the other symbols being the same as in the note in p. 131.

But this complex formula, being divested of those *useless* quantities which express the values of annuities on *two* joint lives, may be reduced to the simple terms alluded to in the text. See the original formula in *Phil. Trans.* for 1791, p. 251; or in Price's *Obs. on Rev. Pay.*, Problem 1, in Note (P).

[3] That is, when B or C is the oldest of the three lives, and not when A is the oldest of the three lives. But, in none of the problems (inserted by him in the *Phil. Trans.* for 1788,

I have thought it proper to make these observations in this place, because the present problem is of considerable importance in enabling us to determine the value of many of the subsequent problems; and is made use of by him for that purpose: therefore the remarks here made will equally apply to those problems in which Mr. Morgan has so used it. Indeed, I believe *there is not a single problem* inserted by him in *any* of his papers in the *Philosophical Transactions*, respecting the value of Contingent Assurances, wherein this prolix and confused method has not been adopted, in order to determine the same.

PROBLEM XXX.[1]

§ 256. To determine the present value of a given sum, payable on the decease of A, provided he be the *second* that fails of three given lives, A, B, C.

SOLUTION.

The sum may be received at the end of the first year, on the happening of either of three different events: 1. that all the three lives fail in that year, A having died second; 2. that A and B fail in the year, A having died last, and that C lives to the end of it; 3. that A and C fail in the year, A having died last, and that B lives to the end of it. The probabilities of the happening of these several events are respectively $\frac{d_x d_y d_z}{3 l_x l_y l_z}$, $\frac{d_x d_y l_{z1}}{2 l_x l_y l_z}$, and $\frac{d_x d_z l_{y1}}{2 l_x l_y l_z}$: which, being added together and multiplied by sv, will give the expectation of receiving the sum at the end of the first year.

But in the second and following years, the given sum may be received on the happening of either of seven different events: 1. that all the three lives become extinct in the year, A having died second; 2. that A and B both fail in the year, A having died last, and that C lives to the end of it;

1789, or 1791) do the values depend on the *seniority* of the lives concerned; for, *either* of the formulæ deduced by him, in the respective problems, will be *equally* correct whether A, B, or C be the oldest life.

[1] Morgan, Prob. 20; and in *Phil. Trans.* for 1791, Prob. 2, p. 253.

3. that A and C both fail in the year, A having died last, and that B lives to the end of it; 4. that A and B both fail in the year, B having died last, and C having failed in either of the preceding years; 5. that A and C both fail in the year, C having died last, and B having failed in either of the preceding years; 6. that only A dies in the year, B living to the end of it, and C having died in either of the preceding years; 7. that only A dies in the year, C living to the end of it, and B having died in either of the preceding years. The probabilities of the happening of these several events in the second year, are respectively $\frac{d_{x1}\,d_{y1}\,d_{z1}}{3l_x\,l_y\,l_z}$, $\frac{d_{x1}\,d_{y1}\,l_{z2}}{2l_x\,l_y\,l_z}$, $\frac{d_{x1}\,d_{z1}\,l_{y2}}{2l_x\,l_y\,l_z}$, $\frac{d_{x1}\,d_{y1}}{2l_x\,l_y}(1-\frac{l_{y1}}{l_y})$, $\frac{d_{x1}\,d_{z1}}{2l_x\,l_z}(1-\frac{l_{y1}}{l_y})$, $\frac{d_{x1}\,l_{y2}}{l_x\,l_y}(1-\frac{l_{z1}}{l_z})$, and $\frac{d_{x1}\,l_{z2}}{l_x\,l_z}(1-\frac{l_{y1}}{l_y})$: which, being added together and multiplied by sv, will give the expectation of receiving the sum at the end of the second year. In like manner it will be found that the probabilities of the happening of these several events in the third year will be respectively denoted by $\frac{d_{x2}\,d_{y2}\,d_{z2}}{3l_x\,l_y\,l_z}$, $\frac{d_{x2}\,d_{y2}\,l_{z3}}{2l_x\,l_y\,l_z}$, $\frac{d_{x2}\,d_{z2}\,l_{y3}}{2l_x\,l_y\,l_z}$, $\frac{d_{x2}\,d_{y2}}{2l_x\,l_y}(1-\frac{l_{z2}}{l_z})$, $\frac{d_{x2}\,d_{z2}}{2l_x\,l_z}(1-\frac{l_{y2}}{l_y})$, $\frac{d_{x2}\,l_{y3}}{l_x\,l_y}(1-\frac{l_{z2}}{l_z})$, and $\frac{d_{x2}\,l_{z3}}{l_x\,l_y}(1-\frac{l_{y2}}{l_y})$: which, being added together and multiplied by sv, will give the expectation of receiving the sum at the end of the third year; and so on for every subsequent year to the utmost extent of human life; the sum of all which yearly expectations will be the total present value of the given sum to be received on the above contingency.

§ 257. Now if these several annual expectations be reduced to their least terms, and then arranged under each other as in the preceding problem, they will be found to form sixteen collateral series; the sum of all which will be equal to s multiplied into $\frac{v(1-ra_{xy})}{2}-\frac{v(1+a_{x1\cdot y})}{2}\cdot\frac{l_{x1}}{l_x}+a_{1x\cdot y}\,\frac{l_{1x}}{2l_x}+\frac{v(1-ra_{xz})}{2}-\frac{v(1+a_{x1\cdot y})}{2}\cdot\frac{l_{x1}}{l_x}+a_{1x\cdot z}\frac{l_{1x}}{2l_x}-\frac{2}{3}v(-ra_{xyz})+\frac{v(1+a_{x1\cdot y\cdot z})}{3}\cdot\frac{2l_{x1}}{l_x}-a_{1x\cdot y\cdot z}\,\frac{2l_{1x}}{3l_x}-\frac{v(1+a_{x\cdot y1\cdot z})}{3}\cdot\frac{l_{y1}}{l_y}+a_{x\cdot 1y\cdot z}\,\frac{l_{1y}}{3l_y}-\frac{v(1+a_{x\cdot y\cdot z1})}{3}\cdot\frac{l_{z1}}{l_z}+a_{x\cdot y\cdot 1z}\,\frac{l_{1z}}{3l_z}$.

But the first three terms here given are, by Prob. XXVII., equal to A_B; the next three terms are, by the second corollary to the same problem, equal to A_C; and the remaining seven terms are, by Prob. XXIX., equal to $-2A_{BC}$. Consequently the total present value of the given sum depending on this contingency will be equal to $s(A_B+A_C-2A_{BC})$.

COROLLARY.

§ 258. If the three lives are equal, or of the same age, then A_B and A_C will each of them become (as in Prob. XXVII., cor. 5) equal to $\frac{v(2-ra_{xx})}{2}$; and A_{BC} will become (as in Prob. XXIX., cor. 2) equal to $\frac{v(1-ra_{xxx})}{3}$. Consequently the present value of the given sum will in such case be represented by $s \times \frac{v}{3}[1-r(3a_{xx}-2a_{xxx})]$, or by one-third of the present value of the given sum payable on the extinction of any two out of the three given lives.[1]

PROBLEM XXXI.[2]

§ 259. To determine the present value of a given sum payable on the decease of A, provided he be the *last* which fails of three given lives, A, B, C.

SOLUTION.

It is evident that the sum can be received at the end of the first year, only on the extinction of all the lives, A having died last, the probability of which is $\frac{d_x\,d_y\,d_z}{3l_x\,l_y\,l_z}$; and which, being multiplied by sv, will give the expectation of receiving the sum at the end of the first year.

But in the second and following years the sum may be received on the happening of either of four different events: 1. that all the lives fail in the year, A having died last; 2. that A and B fail in the year, A having died last, and that C fails in either of the preceding years; 3. that A and C fail in the year, A having died last, and that B fails in either of the preceding years; 4. that only A dies in the year, B and C having failed in either of the preceding years. The probabilities of these several events happening in the second year, are respectively $\frac{d_{x1}\,d_{y1}\,d_{z1}}{3l_x\,l_y\,l_z}$, $\frac{d_{x1}\,d_{y1}}{2l_x\,l_y}(1-\frac{l_{z1}}{l_z})$,

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[1] It will be evident from an inspection of the several quantities, whence this formula is deduced, that a more convenient one might have been formed for the *numerical* solution of the problem: similar to the method pursued in the 39th and subsequent problems. But the present simple formula, at the same time that it is more easily retained in the memory, has also the additional advantage of enabling us readily to discover any error which may arise in the process. I have therefore on this account inserted it: but the reader may, from the values of the several series above given, arrange the formula in such other manner as he may find most convenient to himself. The same observations will apply to many of the subsequent problems in this chapter.

[2] Morgan, Prob. 21; and in *Phil. Trans.* for 1791, Prob. 3, p. 256. Simpson, Prob. 30.

$\frac{d_{x1}\,d_{z1}}{2l_x\,l_y}(1-\frac{l_{y1}}{l_y})$, and $\frac{d_{x1}}{l_x}(1-\frac{l_{y1}}{l_y})(1-\frac{l_{z1}}{l_z})$: which being added together and multiplied by sv, will give the expectation of receiving the sum at the end of the second year

In like manner may be found the expectation of receiving the sum at the end of the third and every subsequent year to the utmost extent of human life: and, if these several yearly expectations be reduced to their lowest terms and arranged under each other, they will form eighteen collateral series, the sum of all which will be the present value required, and which will be found to be equal to s multiplied into $v(1-ra_x)-\frac{v(1-ra_{xy})}{2}+\frac{v(1-a_{x1\cdot y})}{2}\cdot\frac{l_{x1}}{l_x}-a_{1x\cdot y}\frac{l_{1x}}{2l_x}-\frac{v(1-ra_{xz})}{2}+\frac{v(1+a_{x1\cdot z})}{2}\cdot\frac{l_{x1}}{l_x}-a_{1x\cdot z}\frac{l_{1x}}{2l_x}+\frac{v(1-ra_{xyz})}{3}-\frac{v(1+a_{x1\cdot y\cdot z})}{3}\cdot\frac{l_{x1}}{l_x}+a_{1x\cdot y\cdot z}\frac{l_{1x}}{3l_x}+\frac{v(1+a_{x\cdot y1\cdot z})}{6}\cdot\frac{l_{y1}}{l_y}-a_{x\cdot 1y\cdot z}\frac{l_{1y}}{6l_y}+\frac{v(1+a_{x\cdot y\cdot z1})}{6}\cdot\frac{l_{z1}}{l_z}-a_{x\cdot y\cdot 1z}\frac{l_{1z}}{6l_z}$. But this expression is equal to $s\,[v(1-ra_x-A_B-A_C+A_{BC})]$: which therefore denotes the present value required.

COROLLARY.

§ 260. If the three lives are equal, or of the same age A, the present value of the given sum will, agreeably to what has been said in the corollary to the last problem, be denoted by $\frac{sv}{3}[1-r(3a_x-3a_{xx}+a_{xxx})]$, or by one-third of the present value of the given sum to be received on the extinction of the longest of the three lives.

SCHOLIUM.

§ 261. If the present values of the given sum, as found by Prob. XXIX., XXX., and XXXI., be added together, they will be found equal to the present value of the same sum to be received on the decease of A; that is, the sum of those three values will be equal to $sv(1-ra_x)$: which proves the accuracy of the investigations.

PROBLEM XXXII.[1]

§ 262. To determine the present value of a given sum payable on the decease of A, provided he be the *first or second* that fails of three given lives A, B, C.

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[1] Morgan, Prob. 24; and in *Phil. Trans.* for 1791, Prob. 5, p. 261. Simpson's *Sup.* Prob. 40.

SOLUTION.

The given sum may be received at the end of the first year, on the happening of either of four different events: 1. that all the three lives fail in the year, A having died first or second; 2. that A and B both die in the year, and that C lives to the end of it; 3. that A and C both die in the year, and that B lives to the end of it; 4. that only A dies in the year, and that B and C both live to the end of it. The probabilities of the happening of these several events are respectively denoted by $\frac{2d_x d_y d_z}{3l_x l_y l_z}$, $\frac{d_x d_y l_{z1}}{l_x l_y l_z}$, $\frac{d_x d_z l_{y1}}{l_x l_y l_z}$, and $\frac{d_x l_{y1} l_{z1}}{l_x l_y l_z}$: and which being multiplied by sv, will give the expectation of receiving the sum at the end of the first year.

But, in the second and following years, the given sum may be received on the happening of either of eight different events: 1. that all the three lives fail in the year, A having died first or second; 2. that A and B both die in the year, and that C lives to the end of it; 3. that A and C both die in the year, and that B lives to the end of it; 4. that only A dies in the year, and that B and C both live to the end of it; 5. that A dies before B in the year, C having failed in either of the preceding years; 6. that A dies before C in the year, B having failed in either of the preceding years; 7. that only A dies in the year, C having died in either of the preceding years, and B living to the end of it; 8. that only A dies in the year, B having died in either of the preceding years, and C living to the end of it. The probabilities of the happening of these several events in the second year are denoted respectively by $\frac{2d_{x1} d_{y1} d_{z1}}{3l_x l_y l_z}$, $\frac{d_{x1} d_{y1} l_{z2}}{l_x l_y l_z}$, $\frac{d_{x1} d_{z1} l_{y2}}{l_x l_y l_z}$, $\frac{d_{x1} l_{y2} l_{z2}}{l_x l_y l_z}$, $\frac{d_{x1} d_{y1}}{2l_x l_y}(1 - \frac{l_{z1}}{l_z})$, $\frac{d_{x1} d_{z1}}{2l_x l_z}(1 - \frac{l_{y1}}{l_y})$, $\frac{d_{x1} l_{y2}}{l_x l_y}(1 - \frac{l_{z1}}{l_z})$, and $\frac{d_{x1} l_{z2}}{l_x l_z}(1 - \frac{l_{y1}}{l_y})$: which being added together, and multiplied by sv, will give the expectation of receiving the sum at the end of the second year. In like manner, we may find the expectation of receiving the sum at the end of the third and every succeeding year to the utmost extent of human life: the sum of all which expectations will be the total present value of the given sum, to be received on the above contingency.

These several yearly expectations, being reduced to their least terms and arranged under each other, will form sixteen collateral series;[1] the sum of all which is the present value required, and will be found equal to $s(A_B + A_C - A_{BC})$.[1]

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[1] It would be needless for me to swell the present work by inserting the *several* values of these series, since (from the examples which have preceded) they may be readily discovered by an inspection of the resulting formula. Few persons, perhaps, who read this part of the treatise, will be satisfied without going through the several steps of the pro-

COROLLARY.

§ 263. When the three lives are equal, or all of the same age A, this expression (agreeably to what has been said in the corollary to Prob. XXX.) will become equal to $\frac{sv}{3}[2 - r(3a_{xx} - a_{xxx})]$, or equal to the difference between the value of an assurance of the given sum on the two joint lives, and one-third of the value of an assurance of the same sum on the three joint lives.

PROBLEM XXXIII.[2]

§ 264. To determine the present value of a given sum payable on the decease of A, provided he be the *second or third* that fails of three given lives A, B, C.

SOLUTION.

The given sum may be received at the end of the first year, on the happening of either of three different events: 1. that all the three lives become extinct in the year, A having died the second or third; 2. that A and B both fail in the year, A having died last, and that C lives to the end of it; 3. that A and C both fail in the year, A having died last, and that B lives to the end of it. The probabilities of the happening of these several events in the first year, are respectively, $\frac{2d_x d_y d_z}{3l_x l_y l_z}$, $\frac{d_x d_y l_{z1}}{2l_x l_y l_z}$, and $\frac{d_x d_z l_{y1}}{2l_x l_y l_z}$; which being added together and multiplied by sv will give the expectation of receiving the sum at the end of the first year.

But in the second and following years, the sum may be received on the happening of either of eight different events: 1. that all the three lives fail in the year, A having died second or third; 2. that A and B both fail in the year, A having died last, and that C lives to the end of it; 3. that A and C both fail in the year, A having died last, and that B lives to

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cess; in which case, such a measure would not be requisite. And I cannot here avoid the opportunity of recommending the reader to adopt this plan through all the subsequent problems, as he will find it an excellent *praxis* for this species of analysis.

[1] It may be here useful to remark that this formula may be obtained by taking the *sum* of the values deduced from Prob. XXIX. and XXX., or by taking the *difference* between $sv(1 - ra_x)$ and the value deduced from Prob. XXXI., which is evident from the nature of the questions. For the sum is to be received on the death of A provided he dies *first or second*, or (which is the same thing) it is to be received on the death of A provided he does *not* die *third*.

[2] Morgan, Prob. 25; and in *Phil. Trans.* for 1791, Prob. 6, p. 263.

the end of it; 4. that A and B both die in the year, C having died in either of preceding years; 5. that A and C both die in the year, B having died in either of the preceding years; 6. that only A dies in the year, B living to the end of it, and C having died in either of the preceding years; 7. that only A dies in the year, C living to the end of it, and B having died in either of the preceding years; 8. that only A dies in the year, B and C having both failed in either of the preceding years. The probabilities of the happening of these several events in the second year are respectively $\frac{2d_{x1}\,d_{y1}\,d_{z1}}{3l_x\,l_y\,l_z}$, $\frac{d_{x1}\,d_{y1}\,l_{z2}}{2l_x\,l_y\,l_z}$, $\frac{d_{x1}\,d_{z1}\,l_{y2}}{2l_x\,l_y\,l_z}$, $\frac{d_{x1}\,d_{y1}}{l_x\,l_y}(1-\frac{l_{z1}}{l_z})$, $\frac{d_{x1}\,d_{z1}}{l_x\,l_z}(1-\frac{l_{y1}}{l_y})$, $\frac{d_{x1}\,l_{y2}}{l_x\,l_y}(1-\frac{l_{z1}}{l_z})$, $\frac{d_{x1}\,l_{z2}}{l_x\,l_z}(1-\frac{l_{y1}}{l_y})$, and $\frac{d_{x1}}{l_x}(1-\frac{l_{y1}}{l_y})(1-\frac{l_{z1}}{l_z})$: which being added together and multiplied by sv will give the expectation of receiving the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third and every subsequent year to the utmost extent of human life; the sum of all which expectations will be the total present value of the given sum, depending on the above contingency.

These several yearly expectations, being reduced to their least terms and arranged under each other, will form ten collateral series, the sum of all which will be found equal to $s[v(1-ra_x)-A_{BC}]$.[1]

COROLLARY.

§ 265. When all the lives are equal, or of the same age A, the last term in this formula becomes equal to $\frac{v(1-a_{xxx})}{3}$ (agreeably to what has been already said in Prob. XXIX. cor. 2), consequently the whole expression is in this case reduced to $\frac{sv}{3}[2-r(3a_x-a_{xxx})]$: that is, equal to difference between the value of an assurance of the given sum on a single life, and one-third of the value of an assurance of the same sum on the three joint lives.

PROBLEM XXXIV.[2]

§ 266. To determine the present value of a given sum, payable on the decease of A, provided he be the *first or last* that fails of three given lives, A, B, C.

[1] The value of the sum depending on the contingency mentioned in this problem is manifestly equal to the *difference* between the value of such sum to be received on the decease of A, and its value depending on the contingency that A shall be the *first* that fails; or, it is equal to the *sum* of the two values found by Probs. XXX. and XXXI.

[2] Morgan, Prob. 26; *Phil. Trans.* for 1791, Prob. 7, p. 265.

SOLUTION.

The given sum may be received at the end of the first year on the happening of either of four different events: 1. that all the three lives fail in the year, A having died first or last; 2. that A and B both fail in the year, A having died first, and that C lives to the end of it; 3. that A and C both fail in the year, A having died first, and that B lives to the end of it; 4. that only A dies in the year, and that B and C both live to the end of it. The probabilities of the happening of these several events are respectively $\frac{2d_x d_y d_z}{3l_x l_y l_z}$, $\frac{d_x d_y l_{z1}}{2l_x l_y l_z}$, $\frac{d_x d_z l_{y1}}{2l_x l_y l_z}$, and $\frac{d_x l_{y1} l_{z1}}{l_x l_y l_z}$: which being added together and multiplied by sv will give the expectation of receiving the sum at the end of the first year.

But at the end of the second and following years the given sum may be received on the happening of either of seven different events: 1. that all the lives fail in the year, A having died first or last; 2. that A and B both fail in the year, A having died first, and that C lives to the end of it; 3. that A and C both fail in the year, A having died first, and that B lives to the end of it; 4. that only A dies in the year, and that B and C both live to the end of it; 5. that A and B both fail in the year, A having died last, and C having failed in either of the preceding years; 6. that A and C both fail in the year, A having died last, and B having failed in either of the preceding years; 7. that only A dies in the year, B and C having both failed in either of the preceding years. The probabilities of the happening of these several events in the second year, are respectively $\frac{2d_{x1} d_{y1} d_{z1}}{3l_x l_y l_z}$, $\frac{d_{x1} d_{y1} l_{z2}}{2l_x l_y l_z}$, $\frac{d_{x1} d_{z1} l_{y2}}{2l_x l_y l_z}$, $\frac{d_{x1} l_{y2} l_{z2}}{l_x l_y l_z}$, $\frac{d_{x1} d_{y1}}{2l_x l_y}(1-\frac{l_{z1}}{l_z})$, $\frac{d_{x1} d_{z1}}{2l_x l_z} \times (1-\frac{l_{y1}}{l_y})$, and $\frac{d_{x1}}{l_x}(1-\frac{l_{y1}}{l_y})(1-\frac{l_{z1}}{l_z})$: which being added together and multiplied by sv will give the expectation of receiving the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third and every subsequent year to the utmost extent of human life: the sum of all which yearly values will be the total present value of the given sum to be received on the above contingency.

These several yearly expectations, being reduced to their least terms and arranged under each other, will form eighteen collateral series: the sum of all which will be found equal to $s[v(1-ra_x)-A_B-A_C+2A_{BC}]$.[1]

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[1] This formula may be obtained by taking the *sum* of the values deduced from Probs. XXIX. and XXXI., or by taking the *difference* between $sv(1-ra_x)$ and the value deduced from Prob. XXX.

COROLLARY.

§ 267. When the three lives are equal, or of the same age A, this expression (agreeably to what has been said in the corollary to Prob. XXX.) will become equal to $\frac{sv}{3}[2-r(3a_x-3a_{xx}+2a_{xxx})]$: that is equal to the difference between the value of an assurance of the given sum on a single life and on two joint lives, added to two-thirds of the value of an assurance of the same sum on the three joint lives.

SCHOLIUM.

§ 268. If the present values of the given sum, as found by Probs. XXXII., XXXIII., and XXXIV., be added together, they will be found equal to twice the present value of the same sum to be received on the decease of A: that is, the sum of those three values will be equal to $2sv(1-ra_x)$.

PROBLEM XXXV.[1]

§ 269. To determine the present value of a given sum payable on the decease of A or B, provided *either* of them be the *first* that fails of three given lives A, B, C.

SOLUTION.

The given sum may be received at the end of any one year, on the happening of either of six different events: 1. that all the lives fail in the year, A or B having died first; 2. that A and B both die in the year, and that C lives; 3. that A and C both fail in the year, A having died first, and that B lives to the end of it; 4. that B and C both fail in the year, B having died first, and that A lives to the end of it; 5. that only A dies in the year, and that B and C both live to the end of it; 6. that only B dies in the year, and that A and C both live to the end of it. The probabilities of the happening of which several events in the first year are respectively $\frac{2d_x d_y d_z}{3l_x l_y l_z}$, $\frac{d_x d_y l_{z1}}{l_x l_y l_z}$, $\frac{d_x d_z l_{y1}}{2l_x l_y l_z}$, $\frac{d_y d_z l_{x1}}{2l_x l_y l_z}$, $\frac{d_x l_{y1} l_{z1}}{l_x l_y l_z}$, and $\frac{d_y l_{x1} l_{z1}}{l_x l_y l_z}$: which being added together and multiplied by sv will give the expectation of receiving the sum at the end of the first year.

In like manner may be found the expectation of receiving the sum at the end of the second, third, and every subsequent year to the utmost extent of human life: which several yearly values, reduced to their least terms, and being arranged under each other as in Prob. XXIX., will form the following series :—

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[1] *Phil. Trans.* for 1791, Prob. 8, p. 267.

$$sv\left[\frac{4l_x l_y l_z}{l_x l_y l_z}-\frac{4l_{x1} l_{y1} l_{z1}}{l_x l_y l_z}-\frac{l_{x1} l_y l_z}{l_x l_y l_z}+\frac{l_x l_{y1} l_{z1}}{l_x l_y l_z}-\frac{l_x l_{y1} l_z}{l_x l_y l_z}+\frac{l_{x1} l_y l_{z1}}{l_x l_y l_z}+\frac{2l_x l_y l_{z1}}{l_x l_y l_z}-\frac{2l_{x1} l_{y1} l_z}{l_x l_y l_z}\right]+$$

$$sv^2\left[\frac{4l_{x1} l_{y1} l_{z1}}{l_x l_y l_z}-\frac{4l_{x2} l_{y2} l_{z2}}{l_x l_y l_z}-\frac{l_{x2} l_{y1} l_{z1}}{l_x l_y l_z}+\frac{l_{x1} l_{y2} l_{z2}}{l_x l_y l_z}-\frac{l_{x1} l_{y2} l_{z1}}{l_x l_y l_z}+\frac{l_{x2} l_{y1} l_{z2}}{l_x l_y l_z}+\frac{2l_{x1} l_{y1} l_{z2}}{l_x l_y l_z}-\frac{2l_{x2} l_{y2} l_{z1}}{l_x l_y l_z}\right]+$$

$$sv^3\left[\frac{4l_{x2} l_{y2} l_{z2}}{l_x l_y l_z}-\frac{4l_{x3} l_{y3} l_{z3}}{l_x l_y l_z}-\frac{l_{x3} l_{y2} l_{z2}}{l_x l_y l_z}+\frac{l_{x2} l_{y3} l_{z3}}{l_x l_y l_z}-\frac{l_{x2} l_{y3} l_{z2}}{l_x l_y l_z}+\frac{l_{x3} l_{y2} l_{z3}}{l_x l_y l_z}+\frac{2l_{x2} l_{y2} l_{z3}}{l_x l_y l_z}-\frac{2l_{x3} l_{y3} l_{z2}}{l_x l_y l_z}\right]+$$

&c. &c. &c.

the sum of all which will be the total present value required.

§ 270. But, if we compare this general series with the one inserted in page 129, it will be found that the several collateral series of which it is composed are for the most part the same as those which are there given: and that the *only* difference between them is in the common *multiple* and in the *sign* prefixed to the fifth and sixth collateral series. Consequently the sum of these several perpendicular and collateral series may be readily obtained from the process there laid down. For the first and second collateral series will be thus found equal to $\frac{2sv}{3}(1-ra_{xyz})$; the third and fourth equal to $\frac{s}{6l_x}[v(1+a_{x1\cdot y\cdot z})\,l_{x1}-a_{1x\cdot y\cdot z}\,l_{1x}]$; the fifth and sixth equal to $-\frac{s}{6l_y}[v(1+a_{x\cdot y1\cdot z})\,l_{y1}-a_{x\cdot 1y\cdot z}\,l_{1y}]$; and the seventh and eighth equal to $\frac{s}{3l_z}[v(1+a_{x\cdot y\cdot z1})\,l_{z1}-a_{xy\cdot 1z}\,l_{1z}]$. Therefore the total value of the general series above given will be equal to *s* multiplied into $\frac{2v}{3}(1-ra_{xyz})-\frac{1}{6l_x}\times[v(1+a_{x1\cdot y\cdot z})\,l_{x1}-a_{1x\cdot y\cdot z}\,l_{1x}]-\frac{1}{6l_y}[v(1+a_{x\cdot y1\cdot z})\,l_{y1}-a_{x\cdot 1y\cdot z}\,l_{1y}]+\frac{1}{3l_z}\times[v(1+a_{x\cdot y\cdot z1})\,l_{z1}-a_{x\cdot y\cdot 1z}\,l_{1z}]$.

As we shall have occasion to refer again to this formula in some of the following problems, it will be convenient to denote it by a more simple expression; therefore let it be represented by AB_C: that is, let AB_C denote the present value of £1 to be received on the above contingency; consequently $s\times AB_C$ will denote the present value of the given sum under the same circumstances.[1]

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[1] By comparing this formula with the second formula in Prob. XXIX. cor. 1, it will be seen that $AB_C=v(1-ra_{xyz})-C_{AB}$.

COROLLARY.

§ 271. When the three lives are equal, or of the same age A, the last three terms in the above expression destroy each other; and the whole is then reduced to $\frac{2s}{3} \times v(1-ra_{xxx})$: or, to two-thirds of the present value of the given sum payable on the extinction of the three joint lives.

Observations on Mr. Morgan's Method of investigating this Problem.

§ 272. I cannot omit the present opportunity of recalling the reader's attention once more to the singular and confused manner which Mr. Morgan has adopted, in this case also, for finding the value of the general series in page 147: and I am the more induced to do this because it appears that the further Mr. Morgan proceeds in his subject, the more disorder and irregularity seems to run through the whole of his investigations. I would previously, however, entreat the reader's patience whilst I conduct him through this mathematical labyrinth.

The sum of the several expectations of receiving the given sum, on the contingency mentioned in this problem, Mr. Morgan makes equal to the following series:—

$$\begin{aligned}
&sv\left[\frac{2d_x l_y l_z}{3l_x l_y l_z} - \frac{d_x l_{y1} l_z}{6l_x l_y l_z} + \frac{d_x l_y l_{z1}}{3l_x l_y l_z} + \frac{d_x l_{y1} l_{z1}}{6l_x l_y l_z} + \frac{d_y l_z}{2l_y l_z} + \frac{d_y l_{z1}}{2l_y l_z} - \frac{d_x l_y l_z}{2l_x l_y l_z} + \right.\\
&\qquad\left. \frac{d_x l_{y1} l_z}{2l_x l_y l_z} - \frac{d_x l_y l_{z1}}{2l_x l_y l_z} + \frac{d_x l_{y1} l_{z1}}{2l_x l_y l_z}\right] +\\
&sv^2\left[\frac{2d_{x1} l_{y1} l_{z1}}{3l_x l_y l_z} - \frac{d_{x1} l_{y2} l_{z1}}{6l_x l_y l_z} + \frac{d_{x1} l_{y1} l_{z2}}{3l_x l_y l_z} + \frac{d_{x1} l_{y2} l_{z2}}{6l_x l_y l_z} + \frac{d_{y1} l_{z1}}{2l_y l_z} + \frac{d_{y1} l_{z2}}{2l_y l_z} - \right.\\
&\qquad\left. \frac{(l_x - l_{x2}) l_{y1} l_{z1}}{2l_x l_y l_z} + \frac{(l_x - l_{x2}) l_{y2} l_{z1}}{2l_x l_y l_z} - \frac{(l_x - l_{x2}) l_{y1} l_{z2}}{2l_x l_y l_z} + \frac{(l_x - l_{x2}) l_{y2} l_{z2}}{2l_x l_y l_z}\right] +\\
&sv^3\left[\frac{2d_{x2} l_{y2} l_{z2}}{3l_x l_y l_z} - \frac{d_{x2} l_{y3} l_{z2}}{6l_x l_y l_z} + \frac{d_{x2} l_{y2} l_{z3}}{3l_x l_y l_z} + \frac{d_{x2} l_{y3} l_{z3}}{6l_x l_y l_z} + \frac{d_{y2} l_{z2}}{2l_y l_z} + \frac{d_{y2} l_{z3}}{2l_y l_z} - \right.\\
&\qquad\left. \frac{(l_x - l_{x3}) l_{y2} l_{z2}}{2l_x l_y l_z} + \frac{(l_x - l_{x3}) l_{y3} l_{z2}}{2l_x l_y l_z} - \frac{(l_x - l_{x3}) l_{y2} l_{z3}}{2l_x l_y l_z} + \frac{(l_x - l_{x3}) l_{y3} l_{z3}}{2l_x l_y l_z}\right] +\\
&\qquad\qquad \text{\&c.} \qquad \text{\&c.} \qquad \text{\&c.};
\end{aligned}$$

the whole value of which, he says, is equal to s multiplied into

$$\begin{aligned}
&a_{1y\cdot 1z}\frac{l_{1y} l_{1z}}{6l_y l_z} - a_{x\cdot 1y\cdot 1z}\frac{l_{1y} l_{1z}}{6l_y l_z} + a_{y\cdot 1z}\frac{l_{1z}}{3l_z} - a_{xy\cdot 1z}\frac{l_{1z}}{3l_z} - a_{1y\cdot z}\frac{l_{1y}}{6l_y} + a_{x\cdot 1y\cdot z}\frac{l_{1y}}{6l_y} +\\
&\frac{2vr}{3}(a_{yz} - a_{xyz}) + va_{y1\cdot z}\frac{l_{y1}}{6l_y} - va_{x\cdot y1\cdot z}\frac{l_{y1}}{6l_y} - va_{y\cdot z1}\frac{l_{z1}}{3l_z} + va_{xy\cdot z1}\frac{l_{z1}}{3l_z} - va_{y1\cdot z1}
\end{aligned}$$

$$\times \frac{l_{y1} l_{z1}}{6 l_y l_z} + v a_{x \cdot y1 \cdot z1} \frac{l_{y1} l_{z1}}{6 l_y l_z} + \frac{vr}{2} (a_z - a_{yz}) + a_{1z} \frac{l_{1z}}{2 l_z} - a_{y \cdot 1z} \frac{l_{1z}}{2 l_z} - v a_{z1} \frac{l_{z1}}{2 l_z} + v a_{y \cdot z1} \frac{l_{z1}}{2 l_z}.$$[1]

§ 273. Now the last *six* collateral series in the above general expression may be reduced to the *two* following ones, viz.:—

$$\begin{array}{l} sv\left[\dfrac{d_y\, l_{x1}\, l_z}{2l_x\, l_y\, l_z} + \dfrac{d_y\, l_{x1}\, l_{z1}}{2l_x\, l_y\, l_z}\right] + \\ sv^2\left[\dfrac{d_{y1}\, l_{x2}\, l_{z1}}{2l_x\, l_y\, l_z} + \dfrac{d_{y1}\, l_{x2}\, l_{z2}}{2l_x\, l_y\, l_z}\right] + \\ sv^3\left[\dfrac{d_{y2}\, l_{x3}\, l_{z2}}{2l_x\, l_y\, l_z} + \dfrac{d_{y2}\, l_{x3}\, l_{z3}}{2l_x\, l_y\, l_z}\right] + \\ \qquad\quad \&c. \qquad\qquad \&c. \end{array}$$

in which case, that expression, instead of being composed of *twenty*[2] different series, might have been reduced to *twelve*. Or, had the whole of these collateral series been reduced to their simplest terms, they would have formed the *very same* expression as that given in page 147, and which consists of only eight different terms!!! But, owing to this careless and diffuse mode of treating the subject, it follows that the formula, which denotes the value of the given sum depending on the contingency mentioned in this problem, contains (upon Mr. Morgan's method of solution) no less than twenty different terms:[3] and when it is considered that, in order to obtain those terms, it is necessary[4] to sum up between *thirty and forty* different series, the reader is equally struck with surprise and indignation, and is left in doubt of the *motive* which could have induced the author to pursue so anomalous a course.

§ 274. There is, however, still another part of Mr. Morgan's investigation on which I consider myself equally obliged to make some comment. He says that the formula (above alluded to) gives the exact value when C is the oldest of the three lives; "but if A be the oldest, *the symbols must be changed:*" and the value of the given sum he has, in this case, denoted by *another* formula, which (being introduced in a new dress, and in new characters) appears materially different from the former one.[1] It is however (when divested of all its useless and extraneous quantities) precisely the same as that which I have given in page 131, except as to the substitution of $-\frac{1}{3l_x l_y}[v(1+a_{x1\cdot y1\cdot z})\,l_{x1}\,l_{y1} - a_{1x\cdot 1y\cdot z}\,l_{1x}\,l_{1y}]$ for its equal $+\frac{1}{3l_z}[v(1+a_{xy\cdot z1})\,l_{z1} - a_{xy\cdot 1z}\,l_{1z}]$; agreeably to the principles explained in the note in page 131.

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[1] These last five terms are the value of Mr. Morgan's Σ, expressed in his own circuitous manner. The reader will, at first, perhaps hesitate in believing that the *whole* of the above complex expression may be denoted by only *seven* of the terms made use of by Mr. Morgan!!! See the proper formula in § 270.

[2] *Each* of the ten collateral series in p. 148 evidently forms *two* distinct series when expanded by multiplication; the sum of *both* which it is necessary to find.

[3] Lest I should be charged with misrepresentation, I shall here insert Mr. Morgan's formula in his own characters, viz., S into $\frac{x}{3c}\left[\frac{\beta(\mathrm{FK}-\mathrm{AFK})}{2b} + (\mathrm{BK}-\mathrm{ABK})\right] - \frac{\beta(\mathrm{FC}-\mathrm{AFC})}{6b} + \frac{2(r-1)(\mathrm{BC}-\mathrm{ABC})}{3r} + \frac{m(\mathrm{PC}-\mathrm{APC})}{6br} - \frac{d}{3cr}\left[(\mathrm{BT}-\mathrm{ABT}) + \frac{m(\mathrm{PT}-\mathrm{APT})}{2b}\right] + \Sigma$ (Σ denoting the value of S on the contingency of C's surviving B). The other symbols being the same, as already explained in the note in p. 134. See the original formula in *Phil. Trans.* for 1791, p. 268; or in Price's *Obs. on Rev. Pay.*, Note (P), Problem 8.

[4] I mean upon Mr. Morgan's principles, as explained in § 252.

§ 275. It surely cannot be unknown to Mr. Morgan that the value of the given sum does not in this case depend upon the *seniority* of the lives concerned.[2] For, let the ages of A, B, and C be what they may, and let him change his symbols as often as he pleases, he must still come to the *same* general series, at last, as that given in § 270; the sum of which may be expressed in three different ways, according to the method of summing up the several collateral series of which it is composed.[3] And *either* of these formulæ will denote the value of the given sum, whether A, B, or C be the oldest of the three lives.

These observations are the more necessary to be attended to, because the solution to the present problem is of use in enabling us to determine the value of sums depending on the contingencies mentioned in several of the subsequent problems. Consequently, the more simple we render the present formula, the more easily we may obtain the solution to those problems. This motive, therefore, will be a sufficient apology for the present digression.

PROBLEM XXXVI.[4]

§ 276. To determine the present value of a given sum payable on the decease of A or B, provided *either* of them be the *second* that fails of three given lives A, B, C.

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[1] Take Mr. Morgan's own words: "The general rule expressing the value of the reversion will be $=$ S into $\frac{(r-1)(\mathrm{V}-\mathrm{AB})}{r} - \frac{a}{3a}\left[\frac{\beta(\mathrm{HF}-\mathrm{HFC})}{b} + \frac{\mathrm{HB}-\mathrm{HBC}}{2}\right] - \frac{\beta(\mathrm{AF}-\mathrm{AFC})}{6b} + \frac{2(r-1)(\mathrm{AB}-\mathrm{ABC})}{3r} + \frac{m(\mathrm{AP}-\mathrm{APC})}{6br} + \frac{s}{3ar}\left[\frac{(\mathrm{BN}-\mathrm{BNC})}{2} + \frac{m(\mathrm{PN}-\mathrm{PNC})}{b}\right]$." Where V denotes the perpetuity; and the remaining letters, the same quantities as already explained in the note in p. 137. See the original formula in *Phil. Trans.* for 1791, p. 268; and in Price's *Obs. on Rev. Pay.*, Note (P), Problem 8.

[2] This observation will apply to the whole of the problems inserted by Mr. Morgan in the *Phil. Trans.* for 1788, 1789, and 1791. See note 3 in p. 137.

[3] See what has been already said on this subject in § 255, and in note 1 in p. 137.

[4] *Phil. Trans.* for 1791, p. 269.

SOLUTION.

The given sum may be received at the end of the first year on the happening of either of four different events: 1. that all the three lives fail in that year, A or B having died second; 2. that A and B both die in that year, and C lives to the end of it; 3. that A and C both fail in that year, A having died last, and that B lives to the end of it; 4. that B and C both fail in that year, B having died last, and that A lives to the end of it. The probabilities of the happening of these several events are respectively $\frac{2d_x d_y d_z}{3l_x l_y l_z}$, $\frac{d_x d_y l_{z1}}{l_x l_y l_z}$, $\frac{d_x \partial_z l_{y1}}{2l_x l_y l_z}$, and $\frac{d_y d_z l_{x1}}{2l_x l_y l_z}$: which being added together and multiplied by *sv* will give the expectation of receiving the sum at the end of the first year.

But at the end of the second and following years, the sum may be received on the happening of either of eleven different events: 1. that all the lives fail in the year, A or B having died second; 2. that A and B both die in the year, and that C lives to the end of it; 3. that A and C both fail in the year, A having died last, and that B lives to the end of it; 4. that B and C both fail in the year, B having died last, and that A lives to the end of it; 5. that A and B both die in the year, C having died in either of the preceding years; 6. that A and C both fail in the year, A having died last, and B having died in either of the preceding years; 7. that B and C both fail in the year, B having died last, and A having died in either of the preceding years; 8. that only A dies in the year, B living to the end of it, and C having died in either of the preceding years; 9. that only A dies in the year, C living to the end of it, and B having died in either of the preceding years; 10. that only B dies in the year, A living to the end of it, and C having died in either of the preceding years; 11. that only B dies in the year, C living to the end of it, and A having died in either of the preceding years. The probabilities of the happening of these several events in the second year are respectively $\frac{2d_{x1} d_{y1} d_{z1}}{3l_x l_y l_z}$, $\frac{d_{x1} d_{y1} l_{z2}}{l_x l_y l_z}$, $\frac{d_{x1} d_{z1} l_{y2}}{2l_x l_y l_z}$, $\frac{d_{y1} d_{z1} l_{x2}}{2l_x l_y l_z}$, $\frac{d_{x1} d_{y1}}{l_x l_y}(1-\frac{l_{z1}}{l_z})$, $\frac{d_{x1} d_{z1}}{2l_x l_z}(1-\frac{l_{y1}}{l_y})$, $\frac{d_{y1} d_{z1}}{2l_y l_z}(1-\frac{l_{x1}}{l_x})$, $\frac{d_{x1} l_{y2}}{l_x l_y}(1-\frac{l_{z1}}{l_z})$, $\frac{d_x l_{z2}}{l_x l_z}(1-\frac{l_{y1}}{l_y})$, $\frac{d_{y1} l_{x2}}{l_x l_y}(1-\frac{l_{z1}}{l_z})$, and $\frac{d_{y1} l_{z2}}{l_y l_z}(1-\frac{l_{x1}}{l_x})$: which being added together and multiplied by *sv* will give the expectation of receiving the sum at the end of the second year. In like manner may be found the expectation of receiving the sum at the end of the third and every subsequent year to the utmost extent of human life; the sum of all which yearly values will be the total present value of the given sum, to be received on the above contingency.

Now these several yearly expectations, being reduced to their lowest terms, and arranged under each other, will form eighteen collateral series: the sum of all which will be found equal to $s[v(1-ra_{xy})+A_C+B_C-2AB_C]$.

COROLLARY.

§ 277. When all the lives are equal, or of the same age A, this expression (agreeably to what has been said in the corollary to Prob. XXX.) will become equal to $\frac{2sv}{3}[1-r(3a_{xx}-2a_{xxx})]$: or, to two-thirds of the present value of the given sum payable on the extinction of any two out of three given lives.

PROBLEM XXXVII.[1]

§ 278. To determine the present value of a given sum payable on the decease of A or B, provided *either* of them be the *last* that fails of three given lives A, B, C.

SOLUTION.

The given sum can be received at the end of the first year, on the happening of one event only, viz., the extinction of all the lives in that year, C having been the first or second that failed. The probability of this event is $\frac{2d_x\,d_y\,d_z}{3l_x\,l_y\,l_z}$: which being multiplied by sv will give the expectation of receiving the sum at the end of the first year.

But, in the second and following years, the given sum may be received on the happening of either of six different events: 1. that all the three lives fail in the year, C having died first or second; 2. that only A and B fail in the year, C having died in either of the preceding years; 3. that A and C both fail in the year, A having died last, and B having died in either of the preceding years; 4. that B and C both fail in the year, B having died last, and A having died in either of the preceding years; 5. that only A dies in the year, B and C having both failed in either of the preceding years; 6. that only B dies in the year, A and C having both failed in either of the preceding years. The probabilities of the happening of these several events in the second year are respectively $\frac{2d_{x1}\,d_{y1}\,d_{z1}}{3l_x\,l_y\,l_z}$, $\frac{d_{x1}\,d_{y1}}{l_x\,l_y}(1-\frac{l_{z1}}{l_z})$, $\frac{d_{x1}\,d_{z1}}{2l_x\,l_z}(1-\frac{l_{y1}}{l_y})$, $\frac{d_{y1}\,d_{z1}}{2l_y\,l_z}(1-\frac{l_{x1}}{l_x})$, $\frac{d_{x1}}{l_x}(1-\frac{l_{y1}}{l_y})(1-\frac{l_{z1}}{l_z})$, and $\frac{d_{y1}}{l_y}(1-\frac{l_{x1}}{l_x})(1-\frac{l_{z1}}{l_z})$: which being added together

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[1] *Phil. Trans.* for 1791, Prob. 10, p. 272.

and multiplied by sv will give the expectation of receiving the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third year. And so on for all the subsequent years to the utmost extent of human life: the sum of all which yearly values will be the total present value of the given sum, payable on the above contingency.

These several yearly expectations, being reduced to their lowest terms and arranged under each other, will form twenty-two collateral series: the sum of all which will be found equal to $sv[1-r(a_x+a_y-a_{xy})]-s[A_C-B_C+AB_C]$.

COROLLARY.

§ 279. When the three lives are equal, or of the same age A, this expression will become equal to $\frac{2sv}{3}[1-r(3a_x-3a_{xx}+a_{xxx})]$: or, two-thirds of the present value of the given sum, payable on the extinction of the longest of the three lives.

SCHOLIUM.

§ 280. If the present values of the given sum, as found by Probs. XXXV., XXXVI., and XXXVII., be added together, they will be found equal to the present value of the same sum to be received on the decease of A, *added* to the present value of the same sum to be received on the decease of B; that is, the sum of such three values will be equal to $sv[1-ra_x+1-ra_y]$.

PROBLEM XXXVIII.[1]

§ 281. To determine the present value of a given sum, payable on the decease of A or B, provided *either* of them be the *first* or *second* that fails of three given lives, A, B, C.

SOLUTION.

It is evident that in this case the payment of the given sum at the end of any one year depends wholly on the extinction of the joint lives A B in that year, independent of C; its present value therefore will, by Prob. XXII., be in all cases equal to $sv(1-ra_{xy})$.

SCHOLIUM.

§ 282. In the preceding problems I have deduced *correct* expressions for the value of reversionary sums depending on the several contingencies therein mentioned: but the remaining problems, for the most part, involve a contingency for which it is very difficult to find such an expression as will denote the true value of the same, and be likewise fit for general use. The contingency to which I allude is the probability that one life in particular will die before or after another, during any given period of their joint lives. This subject has been already discussed in the fifth chapter, where it is applied to the method of determining the present value of certain reversionary *annuities;* and I now come to consider it again in its application to the method of determining the present value of reversionary *sums*. In the investigation of the following problems, the contingencies above mentioned will be expressed in italics. And since, by means of the two Lemmata in Chapter V., we may obtain a more convenient expression for the expectations of receiving the given sum after the extinction of the oldest life involved, I shall divide the investigation into two distinct parts; the first of which will denote the value of all the expectations for the first m years,[1] or during the continuance of the oldest life involved; and the second will denote the value of all the expectations after that period.

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[1] *Phil. Trans.* for 1800, Prob. 1, p. 22.

PROBLEM XXXIX.[2]

§ 283. To determine the present value of a given sum payable on the decease of A or B, provided *either* of them be the *second* or *third* that fails of three given lives A, B, C.

SOLUTION.

The given sum may be received at the end of the first year on the happening of either of four different events: 1. that all the lives fail in that year; 2. that A and B fail in that year, and that C lives to the end of it; 3. that A dies after C in that year, and that B lives to the end of it; 4. that B dies after C in that year, and that A lives to the end of it. The probabilities of the happening of these several events are respectively $\frac{d_x d_y d_z}{l_x l_y l_z}$, $\frac{d_x d_y l_{z1}}{l_x l_y l_z}$, $\frac{d_x d_z l_{y1}}{2l_x l_y l_z}$, and $\frac{d_y d_z l_{x1}}{2l_x l_y l_z}$: which being added together and multiplied by sv will give the expectation of receiving the sum at the end of the first year.

But in the second and following years, during the existence of the oldest life, the given sum may be received on the happening of either of thirteen different events: 1. that all the lives fail in the year; 2. that A and B fail in the year, and that C lives to the end of it; 3. that A and

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[1] I shall here observe, once for all, that I take m to denote the number of years between the age of the oldest life and that age in the table of observations when human life becomes extinct; consequently the value of m will *vary* in *each* problem according to the seniority of the lives concerned.

[2] *Phil. Trans.* for 1800, Prob. 2, p. 23.

C both fail in the year, A having died last, and that B lives to the end of it; 4. that B and C both fail in the year, B having died last, and that A lives to the end of it; 5. that A and B die in the year, C having failed in either of the preceding years; 6. that A and C die in the year, B having failed in either of the preceding years; 7. that B and C die in the year, A having failed in either of the preceding years; 8. that only A dies in the year, B living to the end of it, and C having died in either of the preceding years; 9. that only A dies in the year, C living to the end of it, and B having died in either of the preceding years; 10. that only B dies in the year, A living to the end of it, and C having died in either of the preceding years; 11. that only B dies in the year, C living to the end of it, and A having died in either of the preceding years; 12. that only A dies in the year, and *that B and C both fail in either of the preceding years, B having died first;* 13. that only B dies in the year, and *that A and C both fail in either of the preceding years, A having died first.* The probabilities of the happening of these several events in the second year, are respectively $\frac{d_{x1}\,d_{y1}\,d_{z1}}{l_x\,l_y\,l_z}$, $\frac{d_{x1}\,d_{y1}\,l_{z2}}{l_x\,l_y\,l_z}$, $\frac{d_{x1}\,d_{z1}\,l_{y2}}{2l_x\,l_y\,l_z}$, $\frac{d_{y1}\,d_{z1}\,l_{x2}}{2l_x\,l_y\,l_z}$, $\frac{d_{x1}\,d_{y1}}{l_x\,l_y}(1-\frac{l_{z1}}{l_z})$, $\frac{d_{x1}\,d_{z1}}{l_x\,l_z}(1-\frac{l_{y1}}{l_y})$, $\frac{d_{y1}\,d_{z1}}{l_y\,l_z}(1-\frac{l_{x1}}{l_x})$, $\frac{d_{x1}\,l_{y2}}{l_x\,l_y}(1-\frac{l_{z1}}{l_z})$, $\frac{d_{x1}\,l_{z2}}{l_x\,l_z}(1-\frac{l_{y1}}{l_y})$, $\frac{d_{y1}\,l_{x2}}{l_x\,l_y}(1-\frac{l_{z1}}{l_z})$, $\frac{d_{y1}\,l_{z2}}{l_y\,l_z}(1-\frac{l_{x1}}{l_x})$, $\frac{d_{x1}}{2l_x}(1-\frac{l_{y1}}{l_y})(1-\frac{l_{z1}}{l_z})$, and $\frac{d_{y1}}{2l_y}\times(1-\frac{l_{x1}}{l_x})(1-\frac{l_{z1}}{l_z})$:[1] which being added together and multiplied by *sv* will give the expectation of receiving the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third and every subsequent year: and, if these several annual expectations be reduced to their lowest terms and arranged under each other, they will form sixteen collateral series; the sum of which, continued to the utmost extent of human life, would be equal to *s* multiplied into $\frac{v(1-ra_x)}{2}+\frac{v(1-ra_y)}{2}-a_{xy}+\frac{v(1+a_{x1\cdot y})}{2}\cdot\frac{l_{x1}}{l_x}+\frac{v(1+a_{x\cdot y1})}{2}\cdot\frac{l_{y1}}{l_y}+\frac{v(1+a_{xz})}{2}-\frac{v(1+a_{x1\cdot z})}{2}\cdot\frac{l_{x1}}{l_x}+\frac{v(1+a_{yz})}{2}-\frac{v(1+a_{y1\cdot z})}{2}\cdot\frac{l_{y1}}{l_y}-v(1-ra_{xyz})-a_{1x\cdot yz}\frac{l_{1x}}{2l_x}-a_{x\cdot 1y\cdot z}\frac{l_{1y}}{2l_y}+a_{xy\cdot 1z}\frac{l_{1z}}{l_z}$. But this expression, independent of the common multiple *s*, may be reduced to $\frac{v}{2}[2-r(a_x+a_y-2a_{xyz})+a_{xz}+a_{yz}]-a_{xy}+\frac{1}{2l_x}[v(a_{x1\cdot y}-a_{x1\cdot z})\,l_{x1}-a_{1x\cdot yz}\,l_{1x}]+\frac{1}{2l_y}[v(a_{x\cdot y1}-a_{y1\cdot z})\,l_{y1}-a_{x\cdot 1y\cdot z}\,l_{1y}]+a_{xy\cdot 1z}\frac{l_{1z}}{l_z}$: and which, for the sake of a more convenient reference, I shall denote by D.

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[1] In these, and all similar cases, throughout the remaining part of this chapter, I suppose it to be an *equal* chance which of the two lives dies before or after the other during the probable term of their joint existence. Now, though this is not strictly correct (and I have on that account pointed out, in § 173, a method of finding an approximate value of such chance in order to determine the value of contingent *Annuities*), yet in the case of *Assurances* the contingencies are so involved that no material error will probably arise in any practical cases by thus indiscriminately supposing the chances to be even. The resulting formulæ are hereby rendered more simple and easy, and are sufficiently accurate for general use.

§ 284. Now, since only part of the several collateral series above mentioned (and which series are represented by the quantities here given) is to be continued to the utmost extent of human life, and as that part will depend on the seniority of the lives concerned, it will be necessary to divide the subsequent investigation of the problem into three distinct cases, according to the seniority of the three lives. For, in each case those several collateral *series* must be continued only *during the existence of the oldest of such lives:* because, *after* that period, we may obtain a more correct value of all the subsequent expectations, by means of the first Lemma in the fifth chapter.

§ 285. *Case* 1. Let A be the oldest of the three lives. In this case, all those collateral series in which the life A is involved must of course be continued to the utmost extent of human life: but all those series in which the life A is not involved must be continued for *n* terms only. Consequently the quantities $\frac{v(1-ra_y)}{2}$, $\frac{v(1+a_{yz})}{2}$, and $\frac{v(1+a_{y1\cdot z})}{2}\cdot\frac{l_{y1}}{l_y}$ will respectively become $\left[\frac{v(1-ra_y)}{2}-\frac{v(1-ra_{y+m})}{2}\times\frac{v^m\,l_{y+m}}{l_y}\right]$, $\left[\frac{v(1+a_{yz})}{2}-\frac{v(1+a_{yz\|m})}{2}\times\frac{v^m\,l_y\,l_{z\|m}}{l_y\,l_z}\right]$ and $\left[\frac{v(1+a_{y1\cdot z})}{2}\cdot\frac{l_{y1}}{l_y}-\frac{v(1+a_{y1\cdot z\|m})}{2}\times\frac{v^m\,l_{y1}\,l_{z\|m}}{l_y\,l_z}\right]$:[1] whence the sum of the first *m* terms of the

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[1] See the method of proceeding in these cases explained in Prob. XXII. cor. 2, and Prob. XXVII. cor. 4. I shall, however, here take the opportunity of noticing an error on this subject, into which Mr. Morgan has inadvertently fallen, and which pervades the two papers inserted by him in 84th and 90th volumes of the *Philosophical Transactions.* In pointing out the method of finding the value of the first *m* terms of the several series above alluded to, in which the oldest life is not involved, he directs us to convert the quantities, expressing the value of annuities on the *whole* life, into similar expressions for the given *term.* This, however, is certainly incorrect; for it is not the *annuities* on those lives which are to be continued for *m* years, but the *series* from which those annuities arise, and which will in most cases make the value of the annuities, resulting therefrom, equal to a term of $m-1$ years only.

Thus, in the investigation of the present problem in the *Philosophical Transactions* (for 1800, Part I. p. 25), he says that, when B is the oldest of the three lives, the annuity on the joint lives A C must be continued for *x* years, whereas it should be continued for $x-1$ years only, as will readily appear, by attending to the steps of the process. The several collateral series above mentioned (that is, the sum of all the expectations for the first m years) will become equal to s multiplied into $\mathsf{D} - \left[\frac{v}{2}(1 - ra_{y+m})\frac{v^m\, l_{y+m}}{l_y} - \frac{v}{2}(1 + a_{yz|m})\frac{v^m\, l_{yz|m}}{l_y\, l_z} + \frac{v}{2}(1 + a_{y1\cdot z|m})\frac{v^m\, l_{y1\cdot z|m}}{l_y\, l_z}\right]$.

But, after the decease of A, the expectations, arising from the several contingencies on which the sum depends, may be more correctly expressed for all the subsequent years by means of the first Lemma in Chapter V. For, since the chance of receiving the sum at the end of any one of those years depends on B's dying in the year, and on A having died before C in either of the preceding years (the probability of which latter contingency is, by the Lemma, denoted by q)[1], it follows that the sum of the expectations for those years, continued to the utmost extent of human life, will be equal to $s.q \times \left[\frac{v^{m+1}(l_{y+m} - l_{y1+m})}{l_y} + \frac{v^{m+2}(l_{y1+m} - l_{y2+m})}{l_y} + \frac{v^{m+3}(l_{y2+m} - l_{y3+m})}{l_y} + \ldots\right]$: an expression which, by Problem XXII. cor. 1, is equal to $sqv(1 - ra_{y+m})\frac{v^m\, l_{y+m}}{l_y}$.[2] Consequently this value,

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annuity on the life A, he also says, should be continued for x years: whereas neither x nor $x-1$ will give the correct value in his formula; for the true value which ought to be *there* substituted for A is $A' + \frac{1}{r-1} \times \frac{q}{ar^x}$. As this error runs through the *whole* of the investigations inserted by Mr. Morgan in the two papers above mentioned, I have thought it necessary to advert to it in this place, in order to explain any difference which might be observed between his formulæ and those which are here given.

I cannot dismiss this subject without censuring the careless manner in which Mr. Morgan's papers in the *Philosophical Transactions* have been printed. Were a proof of this charge required, I could not give a more convincing one than the two formulæ in the page above referred to, and which are too erroneous to be of any public utility. I at first imagined that these mistakes might be *errors of the press;* but, when I observe them faithfully copied into the last edition of Dr. Price's treatise, I can be at no loss in attributing them to their proper source.

[1] Mr. Morgan has given two separate tables for determining the probabilities of survivorship between two lives: the first (which is inserted in *Phil. Trans.* for 1788, p. 387) shows the probability of one life dying *before* another; and the second (which is inserted in *Phil. Trans.* for 1794, p. 229) shows the probability of one life dying *after* the other. They are both given in the last edition of Dr. Price's *Obs. on Rev. Pay.*, vol. i. p. 406, &c.

These tables are deduced from the principles laid down in the first and second Lemma in the fifth chapter of the present work. The *first* table, here alluded to, will occasionally be found useful in determining the values of q and f, which are frequently introduced in the ensuing problems. But as that table is very limited in its extent, I have (in the note in p. 77) laid down a method of approximating to those probabilities, which in most probable cases will be found sufficiently correct.

Mr. Morgan might have saved himself the trouble of calculating the *second* table above alluded to; since its application to any practical cases may always be avoided: and moreover, the values in that table are, in my opinion, incorrectly deduced. It is certain that the solutions to the ensuing problems may be more simply expressed by referring *only* to the first table; as any person will readily discover by comparing the formulæ in the present work with those given by Mr. Morgan. See what has been already observed on this subject in the note in p. 80.

[2] See note 2 in p. 96.

added to the sum of the first m terms of the several collateral series above mentioned, will express the total present value of the given sum in this case required, and which will be found equal to s multiplied into $\mathsf{D}-\frac{v^{m+1}}{l_y}(\frac{1}{2}-q)(1-ra_{y+m})\,l_{y+m}+\frac{l_{z+m}}{2l_z}[(1+a_{yz|m})\,l_{y+m}-(1+a_{y1\cdot z|m})\,l_{y+m}]$.

§ 286. *Case* 2. Let B be the oldest of the three lives. In this case all those collateral series in which the life B is involved must be continued to the utmost extent of human life: but all those series in which the life B is not involved must be continued for m terms only. Consequently, the quantities $\frac{v}{2}(1-ra_x)$, $\frac{v}{2}(1-a_{xz})$, and $\frac{v}{2}(1-\dot{a}_{y1\cdot z})\frac{l_{y1}}{l_y}$ will respectively become equal to $\left[\frac{v}{2}(1-ra_x)-\frac{v}{2}(1-ra_{x+m})\frac{v^m\,l_{x+m}}{l_x}\right]$, $\left[\frac{v}{2}(1+a_{xz})-\frac{v}{2}(1+a_{xz+m})\times\frac{v^m\,l_{xz|m}}{l_x\,l_z}\right]$, &c., $\left[\frac{v}{2}(1+a_{y1\cdot z})\frac{l_{x1}}{l_x}-\frac{v}{2}(1+a_{x1\cdot z|m})\frac{v^m\,l_{x1\cdot z|m}}{l_x\,l_z}\right]$: whence the sum of the first m terms of the several collateral series above mentioned will be equal to s multiplied into $\mathsf{D}-\frac{v}{2}(1-ra_{x+m})\frac{v^m\,l_{x+m}}{l_x}-\frac{v}{2}(1+a_{xz|m})\times\frac{v^m\,l_x\,l_{z|m}}{l_x\,l_z}+\frac{v}{2}(1+a_{x1\cdot z|m})\frac{v^m\,l_{x1\cdot z|m}}{l_x\,l_z}$.

But, after the decease of B, the expectations, arising from the several contingencies on which the sum depends, may be more correctly expressed for all the subsequent years by means of the first Lemma in Chapter V. For, since the chance of receiving the sum at the end of any one of those years depends on A's dying in the year, and on B having died before C in either of the preceding years (the probability of which latter contingency is, by the Lemma, § 151, denoted by f) it follows, from what has been said in the last case, that the sum of all the expectations for those years, continued to the utmost extent of human life, will be equal to $sfv(1-ra_{x+m})\frac{v_m\,l_{x+m}}{l_x}$. Consequently this value, added to the sum of the first m terms of the several collateral series above mentioned, will express the total present value of the given sum in this case required; and which will be found equal to s multiplied into

$$\mathsf{D}-\frac{v^{m+1}}{l_x}(\tfrac{1}{2}-f)(1-ra_{x+m})l_{x+m}+\frac{l_{z+m}}{2l_z}[(1+a_{xz|m})l_{x+m}-(1+a_{x1\cdot z|m})l_{x+m}].$$

§ 287. *Case* 3. Let C be the oldest of the three lives. In this case, all those collateral series in which the life C is involved must be continued to the utmost extent of human life: but all those series in which the life

C is not involved must be continued for m terms only.[1] Consequently the quantities $\frac{v(1-ra_x)}{2}$, $\frac{v(1-ra_y)}{2}\cdot a_{xy}$, $\frac{v(1+a_{x1\cdot y})}{2}\cdot\frac{l_{x1}}{l_x}$, and $\frac{v(1+a_{x\cdot y1})}{2}\cdot\frac{l_{y1}}{l_y}$ will respectively become equal to $\left[\frac{v(1-ra_x)}{2}-\frac{v(1-ra_{x+m})}{2}\cdot\frac{v^m l_{x+m}}{l_x}\right]$, $\left[\frac{v(1-ra_y)}{2}-\frac{v(1-ra_{y+m})}{2}\cdot\frac{v^m l_{y+m}}{l_y}\right]$, $\left[a_{xy}-a_{xy|m}\ \frac{v^m l_x l_{y|m}}{l_x l_y}\right]$, $\left[\frac{v(1-ra_{x1\cdot y})}{2}\cdot\frac{l_{x1}}{l_x}-\frac{v(1+a_{x1\cdot y|m})}{2}\cdot\frac{v^m l_{x1} l_{y|m}}{l_x l_y}\right]$, and $\left[\frac{v(1+a_{x\cdot y1}}{2}\cdot\frac{l_{y1}}{l_y}-\frac{v(1+a_{x\cdot y1|m})}{2}\times\frac{v^m l_x l_{y1|m}}{l_x l_y}\right]$: whence, the sum of the first m terms of the several collateral series above mentioned will be equal to s multiplied into

$$\text{D}-\frac{v(1-ra_x)}{2}\cdot\frac{v^m l_{x+m}}{l_x}-\frac{v(1-ra_{y+m})}{2}\cdot\frac{v^m l_{y+m}}{l_y}+a_{xy|m}\ \frac{v^m l_x l_{y|m}}{l_x l_y}-\frac{v(1+a_{x1\cdot y|m})}{2}\cdot\frac{v^m l_{x1} l_{y|m}}{l_x l_y}-\frac{v(1+a_{x\cdot y1|m})}{2}\cdot\frac{v^m l_x l_{y1|m}}{l_x l_y}.$$

But, after the decease of C, the chance of receiving the given sum at the end of any subsequent year will depend on the happening of either of three events: 1. that *either*[2] A or B dies in the year; 2. that only A dies in the year, B having died before C in either of the preceding years; 3. that only B dies in the year, A having died before C in either of the preceding years. The sum of all the expectations of receiving the given sum on the first contingency, for every subsequent year to the utmost extent of human life, is, by Prob. XXII. cor. 1, equal to $sv(1-ra_{xy|m})\frac{v^m l_x l_{y|m}}{l_x l_y}$;[3] and the sum of all the expectations on the other two contingencies is, as in the two preceding cases, equal to $sfv\,(1-ra_{x+m})\,\frac{v^m l_{x+m}}{l_x}+sqv\,(1-ra_{y+m})\,\frac{v^m l_{y+m}}{l_y}$. Consequently these values, added to the sum of the first m terms of the several collateral series above mentioned, will express the total present value of the given sum in this case required; and which will be found equal to s multiplied into

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[1] Mr. Morgan, in his investigation of these several cases, has followed his general practice of *changing the symbols* in the *last* case, and of introducing *another* formula for the solution thereof. But these new formulæ will always be found to be the *same* as those which he has before inserted, *disguised under different characters*. They arise from the *different* modes of summing up the *same* series: agreeably to what I have before remarked in the notes in pp. 116 and 131.

In the notation adopted by Mr. Morgan, the similarity of the two formulæ is not immediately recognised; and it is possible that he himself was not aware of it. Yet it is singular that he should ever have thought it necessary or useful to recur to *another* expression for the value of the different series, when he had already obtained a formula for that purpose, which (I presume) he must well know was a *general* one. This remark extends to most of the subsequent problems in the present chapter; and will show that, by such acts of supererogation, he has rendered the subject unnecessarily tedious and confused. See the note to the first case of Prob. XLV.

[2] Mr. Morgan says (*Phil. Trans.* for 1800, p. 26) that this chance depends on *both* the lives becoming extinct within the year; which is evidently erroneous.

[3] See note 2 in page 96.

$$\mathsf{D}-\frac{v^{m+1}}{l_x}\Big[(\tfrac{1}{2}-f)(1-ra_{x+m})\,l_{x+m}-\frac{l_{y+m}}{2l_y}\Big((1+a_{xy|m})\,l_{x+m}-$$

$$(1+a_{x1\cdot y|m})\,l_{x1+m}\Big)\Big]$$

$$-\frac{v^{m+1}}{l_y}\Big[(\tfrac{1}{2}-q)(1-ra_{y+m})\,l_{y+m}-\frac{l_{x+m}}{2l_x}\Big((1+a_{xy|m})\,l_{y+m}-$$

$$(1+a_{x\ y1|m})\,l_{y+m}\Big)\Big].$$

COROLLARY.

§ 288. When the three lives are equal, or all of the same age A, the last three quantities in the formula denoted by D vanish entirely; and the general expression for this case will become equal to $sv\times[1-r(a_x+a_{xx}-a_{xxx})]$.

SCHOLIUM.

§ 289. As the expressions deduced from the method of investigation pursued in these three cases will be frequently referred to in the subsequent problems, it will be convenient to denote them by a more simple character. Let us therefore make

$$h=\frac{v^{m+1}}{l_y}\Big[(\tfrac{1}{2}-q)(1-ra_{y+m})\,l_{y+m}+\frac{l_{z+m}}{2l_z}\Big((1+a_{xy|m})\,l_{y+m}-$$

$$(1+a_{x1\cdot z|m})\,l_{y1+m}\Big)\Big]$$

$$k=\frac{v^{m+1}}{l_x}\Big[(\tfrac{1}{2}-f)(1-ra_{x+m})\,l_{x+m}+\frac{l_{z+m}}{2l_z}\Big((1+a_{xz|m})\,l_{x+m}-$$

$$(1+a_{x1\cdot z|m})\,l_{x1+m}\Big)\Big]$$

$$p=\frac{v^{m+1}}{l_y}\Big[(\tfrac{1}{2}-q)(1-ra_{y+m})\,l_{y+m}-\frac{l_{x+m}}{2l_x}\Big((1+a_{xy|m})\,l_{y+m}-$$

$$(1+a_{x\cdot y1|m})\,l_{y1+m}\Big)\Big]$$

$$u=\frac{v^{m+1}}{l_x}\Big[(\tfrac{1}{2}-f)(1-ra_{x+m})\,l_{x+m}-\frac{l_{y+m}}{2l_y}\Big((1+a_{xy|m})\,l_{x+m}-$$

$$(1+a_{x1\cdot y|m})\,l_{x+m}\Big)\Big].$$

Then will the present value of the given sum depending on the contingency mentioned in this problem be respectively expressed by these several formulæ

$$s[\mathsf{D}-h]$$
$$s[\mathsf{D}-k]$$
$$s[\mathsf{D}-p-u];$$

according as A, B, or C is the oldest of the three lives.

PROBLEM XL.[1]

§ 290. To determine the present value of a given sum payable on the decease of A or B, provided *either* of them be the *first* or *last* that fails of three given lives A, B, C.

SOLUTION.

The payment of the given sum at the end of the first year will depend on the happening of either of six different events: 1. that all the lives fail in the year; 2. that A and B fail in the year, and that C lives to the end of it; 3. that A and C fail in the year, A having died first, and that B lives to the end of it; 4. that B and C fail in the year, B having died first, and that A lives to the end of it; 5. that only A dies in the year, and that both B and C live to the end of it; 6. that only B dies in the year, and that both A and C live to the end of it. The probabilities of the happening of these several events in the first year are respectively $\frac{d_x\,d_y\,d_z}{l_x\,l_y\,l_z}$, $\frac{d_x\,d_y\,l_{z1}}{l_x\,l_y\,l_z}$, $\frac{d_x\,d_z\,l_{y1}}{2l_x\,l_y\,l_z}$, $\frac{d_y\,d_z\,l_{x1}}{2l_x\,l_y\,l_z}$, $\frac{d_x\,l_{y1}\,l_{z1}}{l_x\,l_y\,l_z}$, and $\frac{d_y\,l_{x1}\,l_{z1}}{l_x\,l_y\,l_z}$: which being added together and multiplied by sv will give the expectation of receiving the sum at the end of the first year.

But, in the second and following years, the given sum may be received on the happening of either of nine different events: viz., in addition to the six just enumerated, 7. that A and B die in the year, C having failed in either of the preceding years; 8. that only A fails in the year, and *that* B *and* C *both fail in either of the preceding years*, C *having died first;* 9. that only B fails in the year, and *that* A *and* C *both fail in either of the preceding years*, C *having died first*. The probabilities of the happening of these several events in the second year are respectively $\frac{d_{x1}\,d_{y1}\,d_{z1}}{l_x\,l_y\,l_z}$, $\frac{d_{x1}\,d_{y1}\,l_{z2}}{l_x\,l_y\,l_z}$, $\frac{d_{x1}\,d_{z1}\,l_{y2}}{2l_x\,l_y\,l_z}$, $\frac{d_{y1}\,d_{z1}\,l_{x2}}{2l_x\,l_y\,l_z}$, $\frac{d_{x1}\,l_{y2}\,l_{z2}}{l_x\,l_y\,l_z}$, $\frac{d_{y1}\,l_{x2}\,l_{z2}}{l_x\,l_y\,l_z}$, $\frac{d_{x1}\,d_{y1}}{l_x\,l_y}(1-\frac{l_{z1}}{l_z})$, $\frac{d_{x1}}{2l_x}(1-\frac{l_{y1}}{l_y})(1-\frac{l_{z1}}{l_z})$, and $\frac{d_{y1}}{2l_y}(1-\frac{l_{x1}}{l_x})(1-\frac{l_{z1}}{l_z})$: which being added

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[1] *Phil. Trans.* for 1800, Prob. 3, p. 28.

together and multiplied by sv will give the expectation of receiving the sum at the end of the second year. In like manner, we may find the expectation of receiving the sum at the end of the third and every subsequent year: and, if these several annual expectations be reduced to their lowest terms, and arranged under each other, they will form sixteen collateral series, the sum of which, continued to the utmost extent of human life, would be equal to s multiplied into $\frac{v(1-ra_x)}{2}+\frac{v(1-ra_y)}{2}+a_{xy}-\frac{v(1-a_{x1\cdot y})}{2}\cdot\frac{l_{x1}}{l_x}-\frac{v(1+a_{x\cdot y1})}{2}\cdot\frac{l_{y1}}{2l_y}-\frac{v(1+a_{xz})}{2}+\frac{v(1+a_{x1\cdot z})}{2}\cdot\frac{l_{x1}}{l_x}-\frac{v(1+a_{yz})}{2}+\frac{v(1+a_{y1\cdot z})}{2}\cdot\frac{l_{y1}}{l_y}+v(1-ra_{xyz})+\frac{a_{1x\cdot yz}l_{1x}}{2l_x}+a_{x\cdot 1y\cdot z}\frac{l_{1y}}{2l_y}-a_{xy\cdot 1z}\times\frac{l_{1z}}{l_z}$: an expression which, independent of the common multiple s, may be reduced to the formula $\frac{v}{2}[2-r(a_x+a_y+2a_{xyz})-a_{xz}-a_{yz}]+a_{xy}-\frac{1}{2l_x}\times[v(a_{x1\cdot y}-a_{x1\cdot z})l_{x1}-a_{1x\cdot yz}l_{1x}]-\frac{1}{2l_y}[v(a_{x\cdot y1}-a_{y1\cdot z})\,l_{y1}-a_{x\cdot 1y\cdot z}\,l_{1y}]-a_{xy\cdot 1z}\times\frac{l_{1z}}{2l_z}$; and which, for the sake of a more convenient reference, I shall denote by **E**.

§ 291. *Case* 1. Let A be the oldest of the three lives. In this case, the sum of the first m terms of the several collateral series above mentioned will, by pursuing the same steps as in the first case of the preceding problem, be equal to $\mathbf{E}-\frac{v(1-ra_{y+m})}{2}\cdot\frac{v^m l_{y+z}}{l_y}+\frac{v(1+a_{xy|m})}{2}\cdot\frac{v^m l_y\, l_{z|m}}{l_y\, l_z}-\frac{v(1+a_{y1\cdot z|m})}{2}\cdot\frac{v^m\, l_{y1}\, l_{z|m}}{l_y\, l_z}$. After the decease of A, however, the expectations arising from these contingencies may be more correctly expressed for all the subsequent years by means of the second Lemma in the fifth chapter; for, since the chance of receiving the sum at the end of any one of those years depends on B's dying in the year, and on A having died after C in either of the preceding years (the probability of which latter contingency is, by the Lemma, denoted by $1-q$), we shall find that the sum of the expectations for those years, continued to the utmost extent of human life, will be equal to $s(1-q)\times v(1-ra_{y+m})\,\frac{v^m l_{y+m}}{l_y}$. Consequently this value, added to the sum of the first m terms of the several collateral series above mentioned, will make the total present value of the given sum to be equal to $s\,[\mathbf{E}+h]$.

§ 292. *Case* 2. Let B be the oldest of the three lives. In this case, the sum of the first m terms of the several collateral series above mentioned will, by pursuing the same steps as in the second case to the preceding problem, be now equal to $\mathbf{E}-\frac{v(1-ra_{x+m})}{2}\cdot\frac{v^m l_{x+m}}{l_x}-\frac{v(1+a_{xz\|m})}{2}\cdot\frac{v^m l_{xz\|m}}{l_x l_z}-\frac{v(1+a_{x1\cdot z})}{2}\cdot\frac{v^m l_{x1\cdot z\|m}}{l_x l_z}$. After the decease of B, however, the expectation arising from these contingencies may be more correctly expressed for all the subsequent years by means of the Lemma above mentioned; for, since the chance of receiving the sum at the end of any one of those years depends on A's dying in the year, and on B having died after C in either of the preceding years (the probability of which is, by the Lemma, denoted by $1-f$), we shall find that the sum of the expectations for those years, continued to the utmost extent of human life, will be equal to $s.(1-f)\times v(1-ra_{x+m})\frac{v^m l_{x+m}}{l_x}$. Consequently, this value added to the sum of the first m terms as above found, will express the total present value of the given sum in this case required. Whence, such present value will be equal to $s[\mathbf{E}+k]$.

§ 293. *Case* 3. Let C be the oldest of the three lives. In this case, the sum of the first m terms of the several collateral series above mentioned, will (agreeably to the method pursued in the third case to the preceding problem) be equal to $\mathbf{E}-\frac{v(1-ra_{x+m})}{2}\cdot\frac{v^m l_{x+m}}{l_x}-\frac{v(1-ra_{y+m})}{2}\cdot\frac{v^m l_{y+m}}{l_y}-\frac{v^m a_{xy\|m} l_x l_{y\|m}}{l_x l_y}+\frac{v(1+a_{x1\cdot y\|m})}{2}\cdot\frac{v^m l_{x1\cdot y}}{l_x l_y}+\frac{v(1+a_{x\cdot y1\|m})}{2}\cdot\frac{v^m l_{x\cdot y1}}{l_x l_y}$. But, after the decease of C, the expectations arising from these contingencies may be more correctly expressed for all the subsequent years by means of the Lemma above mentioned; for, the chance of receiving the sum at the end of any one of those years will depend on the happening of either of three events: 1. that A and B both fail in the year; 2. that A fails in the year, B having died after C in either of the preceding years; 3. that B fails in the year, A having died after C in either of the preceding years. The probabilities of the happening of these three events in the $(m+1)$st year, are respectively $\frac{d_{x+m}d_{y+m}}{l_x l_y}$, $\frac{d_{x+m}}{l_x}(1-f-\frac{l_{y+m}}{l_y})$ and $\frac{d_{y+m}}{l_y}(1-q-\frac{l_{x+m}}{l_x})$:[1] which being added together and multiplied by v^{m+1}, will give the expectation of receiving the sum at the end of the $(m+1)$st year.

In like manner, the probabilities of the happening of these events in the $(m+2)$nd year are respectively $\frac{d_{x1+m}d_{y1+m}}{l_x l_y}$, $\frac{d_{x1+m}}{l_x}(1-f-\frac{l_{y1+m}}{l_y})$ and

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[1] See the Scholium in p. 81.

$\frac{d_{y1+m}}{l_y}(1-q-\frac{l_{x1+m}}{l_x})$: which being added together and multiplied by v^{m+2} will give the expectation of receiving the sum at the end of the $(m+2)$nd year. By the same method of reasoning we may find the expectation of receiving the sum at the end of the $(m+3)$rd year: and so on to the utmost extent of human life; the sum of all which annual expectations, when reduced to their lowest terms, will be found equal to s multiplied into $(1-f)\ v(1-ra_{x+m})\frac{v^m\ l_{x+m}}{l_x}+(1-q)\ v(1-ra_{y+m})\frac{v^m\ l_{y+m}}{l_y}-v(1+a_{xy|m})$ $\frac{v^m\ l_x\ l_{y|m}}{l_x\ l_y}+a_{xy|m}\frac{v^m\ l_x\ l_{y|m}}{l_x\ l_y}$: which being added to the sum of the first m terms of the several collateral series above mentioned will express the total present value of the given sum in this case required. Whence, such present value will be equal to $s[\mathrm{E}+p+u]$.

COROLLARY.

§ 294. When all the lives are equal, or of the same age A, the last three quantities in the formula denoted by E vanish entirely; and the general expression for this case will become $sv[1-r(a_x-a_{xx}+a_{xxx})]$.

SCHOLIUM.

§ 295. Since it appears that the present value of the given sum, payable on the contingency mentioned in this problem, added to the present value of the same sum, payable on the contingency mentioned in the preceding problem, are together equal to $s\times v[2-r(a_x+a_y)]$; it follows, that the value of one of them being determined, the other may be easily obtained by subtracting such value from the general expression here given; provided the ages of the lives are the same in both instances.

PROBLEM XLI.[1]

§ 296. To determine the present value of a given sum, payable on the decease of A, provided he be the *second* that dies of three given lives, A, B, C, and provided C dies before B.[2]

SOLUTION.

The chance of receiving the given sum at the end of the first year will depend on the happening of either of two events: 1. that all the lives fail in that year, C having died first and A next; 2. that A and C both fail in that year, A having died last, and that B lives to the end of it. The probabilities of the happening of these two events are respectively $\frac{d_x d_y d_z}{6 l_x l_y l_z}$, and $\frac{d_x d_z l_{y1}}{2 l_x l_y l_z}$: which being added together and multiplied by *sv* will give the expectation of receiving the sum at the end of the first year.

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[1] Simpson's *Sup.*, Prob. 38, and Prob. 4, in p. 74. Dodson, vol. iii. Ques. 42 to 47. Morgan, Prob. 18; and in *Phil. Trans.* for 1789, p. 41.

[2] That is, provided C dies first, and A second of the three lives; or, provided C be then dead and B living: but I have thought it best, for the sake of uniformity, to preserve the mode of expression adopted in the text.

But the chance of receiving the given sum at the end of the second and following years will depend on the happening of either of four different events: 1. that all the lives fail in the year, C having died first and A next; 2. that A and C both fail in the year, A having died last, and that B lives to the end of it; 3. that both A and B fail in the year, A having died first, and that C dies in either of the preceding years; 4. that only A dies in the year, B living to the end of it, and C having died in either of the preceding years. The probabilities of the happening of these several events in the second year are respectively $\frac{d_{x1} d_{y1} d_{z1}}{6 l_x l_y l_z}$, $\frac{d_{x1} d_{z1} l_{y2}}{2 l_x l_y l_z}$, $\frac{d_{x1} d_{y1}}{2 l_x l_y}(1-\frac{l_{z1}}{l_z})$, and $\frac{d_{x1} l_{y2}}{l_x l_y}(1-\frac{l_{z1}}{l_z})$: which being added together and multiplied by *sv* will give the expectation of receiving the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third and every subsequent year, to the utmost extent of human life; the sum of all which yearly values will be the total present value of the given sum to be received on the above contingency.

These several yearly expectations being reduced to their least terms, and arranged under each other, will form twelve collateral series: the sum of all which, or the present value required, will be found equal to $s(A_B - A_{BC})$.[1]

COROLLARY.

§ 297. When the lives are all equal, or of the same age A, this ex-

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[1] This problem is one of frequent occurrence, and is the first of that series of problems, inserted by Mr. Morgan in the *Philosophical Transactions,* for determining the value of contingent reversions where three lives are involved in the survivorship. Mr. Morgan has, in his usual manner, given *two* formula for the solution of the problem; leading us thereby to imagine that the value of the given sum depends on the *seniority* of the lives concerned. This, however, is not the fact; for the theorem which I have given in the text is a *general* one, and will apply to either case.

As it may be satisfactory to the reader to see the different results which are obtained from the present investigation and from that pursued by Mr. Morgan, I shall here subjoin his formula. But since the same lives are not involved in the same contingencies, I shall (in order to prevent any misunderstanding on this subject) prefix his own statement of the problem.

"To determine the value of a given sum payable on the contingency of C's surviving B, provided the life of A shall be then extinct.

"When either B or C is the oldest of the three lives, the value of the given sum will pression (agreeably to what has been said in the corollary to Prob. XXX.) will become equal to $\frac{sv}{6}[1-r(3a_{xx}-2a_{xxx})]$; that is, equal to one-sixth of the present value of the given sum payable on the extinction of any two out of the three given lives.

PROBLEM XLII.[1]

§ 298. To determine the present value of a given sum payable on the decease of A, provided he be the *last* that dies of three given lives A, B, C, and provided C dies before B.[2]

SOLUTION.

The given sum cannot be received at the end of the first year unless all the three lives fail in that year, and in the order above mentioned; the probability of which is $\frac{d_x\,d_y\,d_z}{6l_x\,l_y\,l_z}$: which being multiplied by sv will give the expectation of receiving the sum at the end of the first year.

But, in the second and following years, the given sum may be received on the happening of either of three different events: 1. that all the lives fail in the year, in the order above mentioned; 2. that A and B both fail in the year, A having died last, and C having failed in either of the pre-

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be = S into $\frac{\kappa}{6c}\times\left[\frac{\beta(\text{FK}-\text{AFK})}{b}-(\text{BK}-\text{ABK})\right]+\frac{\beta}{3b}\times(\text{FC}-\text{AFC})-\frac{r-1}{3r}\times(\text{BC}-\text{ABC})-\frac{m}{3br}\times(\text{PC}-\text{APC})+\frac{d}{6cr}\times\left[(\text{BT}-\text{ABT})-\frac{m(\text{PT}-\text{APT})}{b}\right]$.

"When A is the oldest of the three lives, the value will be = S into $\Sigma-\frac{\kappa}{3c}\times\left[\frac{(a\text{HK}-\text{HBK})}{a}+\frac{\text{AK}-\text{ABK}}{2}\right]-\frac{a}{6a}\times(\text{HC}-\text{HBC})-\frac{r-1}{3r}\times(\text{AC}-\text{ABC})+\frac{s}{6ar}\times(\text{NC}-\text{NBC})+\frac{d}{3cr}\times\left[\frac{\text{AT}-\text{ABT}}{2}+\frac{s(\text{NT}-\text{NBT})}{a}\right]$:" where Σ denotes the same as B_c in the notation used in the present treatise, and the other characters the same as already explained in the notes in pp. 133 and 136.

These formulæ are taken from the last edition of Dr. Price's *Obs. on Rev. Pay.*, vol. i. p. 392: the gross and careless errors therein being first corrected. It is hardly necessary to remark (after the repeated observations that I have made of a similar kind) that *all* those expressions in the second formula, and some of those in the first, which involve *two* joint lives are totally useless; since they might be made to vanish, in the manner pointed out in p. 135. They consequently now only tend to make the arithmetical solution unnecessarily tedious and confused: and the reader perhaps may be surprised to learn that these two formulæ are precisely the same, disguised under different symbols!!!

[1] Morgan, Prob. 29; and in *Phil. Trans.* for 1794, Prob. 6, p. 253.

[2] That is, provided C dies first, B next, and A last.

ceding years; 3. that only A dies in the year, and *that both B and C fail in either of the preceding years, C having died first.* The probabilities of the happening of these several events in the second year are respectively $\frac{d_{x1}\,d_{y1}\,d_{z1}}{6l_x\,l_y\,l_z}$, $\frac{d_{x1}\,d_{y1}}{2l_x\,l_y}(1-\frac{l_{z1}}{l_z})$, and $\frac{d_{x1}}{2l_x}(1-\frac{l_{y1}}{l_y})(1-\frac{l_{z1}}{l_z})$: which being added together and multiplied by *sv* will give the expectation of receiving the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third and every subsequent year during the continuance of the oldest of the given lives; and if these several annual expectations be reduced to their lowest terms and arranged under each other, they will form fourteen collateral series, the first m terms of which will vary according to the seniority of the lives concerned.

§ 299. *Case* 1. Let A be the oldest of the three lives. In this case the several series here alluded to must be continued to the utmost extent of human life; because the life A is involved in each collateral series. Therefore the value of such series will be equal to s multiplied into $\frac{v(1-ra_x)}{2}+\frac{a_{xy}}{2}-a_{1x\cdot y}\frac{l_{1x}}{2l_x}-\frac{v(1+a_{xz})}{2}+\frac{v(1+a_{x1\cdot z})}{2}\cdot\frac{l_{x1}}{l_x}+\frac{v(1-ra_{xyz})}{6}-\frac{v(1+a_{x1\cdot yz})}{6}\times\frac{l_{x1}}{l_x}+a_{1x\ yz}\frac{l_{1x}}{6l_x}+\frac{v(1+a_{x\cdot y1\cdot z})}{3}\cdot\frac{l_{y1}}{l_y}+a_{x\cdot 1y\cdot z}\frac{l_{1y}}{6l_y}-\frac{v(1+a_{xy\cdot z1})}{6}\cdot\frac{l_{z1}}{l_z}-a_{xy\cdot 1z}\frac{l_{1z}}{3l_z}$: an expression which, independent of the common multiple s, may be reduced to $\frac{v}{6}[1-r(3a_x+a_{xyz})-3a_{xz}]+\frac{a_{xy}}{2}+\frac{1}{6l_x}[v(2+3a_{x1\cdot z}-a_{x1\cdot yz})\,l_{x1}-(3a_{1x\cdot y}-a_{1x\cdot yz})\,l_{1x}]+\frac{1}{6l_y}[v(1+a_{x\cdot y1\cdot z})2l_{y1}+a_{x\cdot 1y\cdot z}\,l_{1y}]-\frac{1}{6l_z}[v(1+a_{xy\cdot z1})\,l_{z1}+a_{xy\cdot 1z}\,2l_{1z}]$; and which I shall denote by **F**. Therefore, when A is the oldest of the three lives, the present value of the given sum will be represented by $s\times$**F**.

§ 300. *Case* 2. Let B be the oldest of the three lives. In this case the sum of the first m terms of the several collateral series above mentioned will be found equal to $\mathbf{F}-\frac{v(1-ra_{x+m})}{2}\cdot\frac{v^m\,l_{x+m}}{l_x}+v(1+a_{xz|m})\frac{v^m l_x\,l_{z|m}}{l_x\,l_z}-\frac{v(1+a_{x1\cdot z|m})}{2}\cdot\frac{v^m\,l_{x1}\,l_{z|m}}{l_x\,l_z}$. But, after the extinction of the life of B, the sum of the expectations for all the subsequent years may be more correctly expressed as in the second case of Prob. XL. by $s(1-f)\times v(1-ra_{x+m})\frac{v^m\,l_{x+m}}{l_x}$. Therefore, if this value be added to the sum of the first m terms of the several series just found, it will make the total present value of the given sum, when B is the oldest of the three lives, equal to $s[\mathbf{F}+k]$.

§ 301. *Case* 3. Let C be the oldest of the three lives. In this case, the sum of the first m terms of the several collateral series above alluded to will be found equal to $\mathbf{F}-\frac{v(1-ra_{x+m})}{2}\cdot\frac{l_{x+m}\,v^m}{l_x}-a_{xy}\,\frac{l_x\,l_{y|m}\,v^m}{2l_x\,l_y}+a_{1x\cdot y|m}\frac{l_{1x}\,l_{y|m}\,v^m}{2l_x\,l_y}$. But, after the extinction of the life of C, the chance of receiving the given sum at the end of any one year will depend on the happening of two events only: 1. that A and B both fail in the year, A having died last; 2. that only A fails in the year, B having died after C in either of the preceding years: the probabilities of the happening of these events in the $(m+1)$st year are respectively $\frac{d_x\,d_{y|m}}{2l_x\,l_y}$ and $\frac{d_{x+m}}{l_x}\,(1-f-\frac{l_{y+m}}{l_y})$, which being multiplied by v^{m+1} will give the expectation of receiving the sum at the end of the $(m+1)$st year. In like manner we might proceed to find the expectations of receiving the given sum for every subsequent year to the utmost extent of human life; and the sum of all those expectations will be found equal to s multiplied into $(1-f)\,v(1-ra_{x+m})\,\frac{l_{x+m}\,v^m}{l_x}-\frac{v(1+a_{xy|m})}{2}\times\frac{l_x\,l_{y\|m}}{l_x\,l_y}+a_{xy|m}\frac{l_x\,l_{y|m}\,v^m}{2l_x\,l_y}+\frac{v(1+a_{x1\cdot y|m})}{2}\cdot\frac{l_{x1}\,l_{y|m}\,v^m}{l_x\,l_y}-a_{1x\cdot y}\frac{l_{1x\cdot y|m}\,v^m}{2l_x\,l_y}$. Consequently this value, added to the sum of the first m terms of the several collateral series above mentioned, will express the total present value of the given sum, in this case required. Whence, such present value will be equal to $s[\mathbf{F}+u]$.[1]

COROLLARY.

§ 302. When the three lives are equal, or all of the same age A, the last three quantities in the formulæ denoted by $\mathbf{F}$ vanish altogether, and the general expression in this case will become $\frac{sv}{6}[1-r(3a_x-3a_{xx}+a_{xxx})]$; that is, equal to one-sixth of the present value of the given sum, payable on the extinction of the longest of the three lives.

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[1] Mr. Morgan has divided this problem (I know not for what reason) into *four* distinct cases; and in his investigations of these cases he has fallen into the same errors and absurdities that I have so frequently noticed in the preceding pages. But as it is not my wish to swell the present work with those disgusting repetitions, and as the remarks which I have already made apply with nearly equal propriety to *all* Mr. Morgan's problems, I shall not here enter more into detail. It may however be useful, perhaps, for Mr. Morgan to know that it is impossible that L (or the value of an annuity on the longest of the three lives) can properly arise in *any* of his problems, unless those lives are equal.

PROBLEM XLIII.[1]

§ 303. To determine the present value of a given sum payable on the decease of A, provided he be the *first* or *second* that fails of three given lives A, B, C; and provided C, in the latter case, dies before B.

SOLUTION.

The payment of the sum at the end of the first year will depend on the happening of either of four different events: 1. that all the lives fail in the year, A having died first, or C having died first and A second; 2. that A and B fail in the year, A having died first and that C lives to the end of it; 3. that A and C die in the year, and that B lives to the end of it; 4. that only A dies in the year, and that B and C both live to the end of it. The probabilities of the happening of these several events are respectively $\frac{d_x\, d_y\, d_z}{2l_x\, l_y\, l_z}$, $\frac{d_x\, d_y\, l_{z1}}{2l_x\, l_y\, l_z}$, $\frac{d_x\, d_z\, l_{y1}}{l_x\, l_y\, l_z}$, and $\frac{d_x\, l_{y1}\, l_{z1}}{l_x\, l_y\, l_z}$: which being added together and multiplied by sv will give the expectation of receiving the sum at the end of the first year.

But, in the second and following years, the given sum may be received on the happening of either of six different events: viz., in addition to the four already enumerated, 5. that A and B fail in the year, A having died first, and that C dies in either of the preceding years; 6. that only A dies in the year, B living to the end of it, and C having died in either of the preceding years. The probabilities of the happening of these six different events in the second year are respectively $\frac{d_x\, a_y\, d_{z|1}}{2l_x\, l_y\, l_z}$, $\frac{d_x d_{y|1}\, l_{z2}}{2l_x\, l_y\, l_z}$, $\frac{d_x d_{z|1}\, l_{y2}}{l_x\, l_y\, l_z}$, $\frac{d_{x1}\, l_{y2}\, l_{z2}}{l_x\, l_y\, l_z}$, $\frac{d_{x1}\, d_{y1}}{2l_x\, l_y}(1-\frac{l_{x1}}{l_z})$, and $\frac{d_{x1}\, l_{y2}}{l_x\, l_y}(1-\frac{l_{z1}}{l_z})$: which being added together and multiplied by sv^2 will give the expectation of receiving the sum at the end of the second year. And so for every subsequent year to the utmost extent of human life: the sum of all which values will be the total present value of the sum required; and which will be found equal to $s \times A_B$, or to the present value of the given sum payable on the decease of A, provided he be the first that dies of the two lives A, B, as found by Prob. XXVII. The truth of which is evident: for the payment of the given sum can be prevented only by the event of B dying before A.[2]

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[1] *Phil. Trans.* for 1800, Prob. 4, p. 32.

[2] It may be here useful to remark that the present value found by this problem is equal to the sum of the two values found by Prob. XXIX. and XLI.: and their agreement in this particular confirms the accuracy of the investigation.

PROBLEM XLIV.[1]

§ 304. To determine the present value of a given sum, payable on the decease of A, provided he be the *second* or *third* that fails of three given lives A, B, C; and provided C dies before B.

SOLUTION.

The given sum may be received at the end of the first year, on the happening of either of two events: 1. that all the lives become extinct in that year, C having died first; 2. that A and C both fail in that year, C having died first, and that B lives to the end of it. The probabilities of the happening of these events are respectively $\frac{d_x d_y d_z}{3l_x l_y l_z}$, and $\frac{d_x d_z l_{y1}}{2l_x l_y l_z}$: which being added together and multiplied by *sv* will give the expectation of receiving the sum at the end of the first year.

But, in the second and following years, the given sum may be received on the happening of either of five different events: 1. that all the lives fail in the year, C having died first; 2. that A and C both fail in the year, C having died first, and that B lives to the end of it; 3. that A and B both die in the year, C having died in either of the preceding years; 4. that only A dies in the year, B living to the end of it, and C having died in either of the preceding years; 5. that only A fails in the year, and *that B and C both fail in either of the preceding years, C having died first.* The probabilities of the happening of these several events in the second year are respectively $\frac{d_x d_y d_{z|1}}{3l_x l_y l_z}$, $\frac{d_{x1} d_{z1} l_{y2}}{2l_x l_y l_z}$, $\frac{d_{x1} d_{y1}}{l_x l_y}(1-\frac{l_{z1}}{l_z})$, $\frac{d_{x1} l_{y2}}{l_x l_y}(1-\frac{l_{z1}}{l_z})$, and $\frac{d_{x1}}{2l_x}(1-\frac{l_{y1}}{l_y})\ (1-\frac{l_{z1}}{l_z}$: which being added together and multiplied by sv^2 will give the expectation of receiving the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third and every succeeding year: and if these several annual expectations be reduced to their lowest terms and arranged under each other, they will form fourteen collateral series; the sum of the first *m* terms of which will vary according to the seniority of the lives concerned.

§ 305. *Case* 1. Let A be the oldest of the three lives. In this case, the terms of the several series here alluded to must be continued to the utmost extent of human life, because the life of A is involved in each collateral series: therefore the sum of it will be equal to *s* multiplied into

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[1] *Phil. Trans.* for 1800, Prob. 5, p. 33.

$\frac{v(1-ra_x)}{2}+\frac{v(1+a_{xy})}{2}-\frac{v(1+a_{x1\cdot y})}{2}\cdot\frac{l_{x1}}{l_x}-\frac{v(1+a_{xz})}{2}+\frac{v(1+a_{x1\cdot z})}{2}\cdot\frac{l_{x1}}{l_x}-\frac{v(1-ra_{xyz})}{6}+\frac{v(1+a_{x1\cdot yz})}{6}\cdot\frac{l_{x1}}{l_x}-a_{1x\cdot yz}\frac{l_{1x}}{6l_x}+\frac{v(1+a_{y1\cdot z})}{6}\cdot\frac{l_{y1}}{l_y}+a_{x\cdot 1y\cdot z}\frac{l_{1y}}{3l_y}-\frac{v(1+a_{xy\cdot z1})}{3}\cdot\frac{l_{z1}}{l_z}-a_{xy\cdot 1z}\frac{l_{1z}}{6l_z}$: an expression which, independent of the common multiple s, may be reduced, as in the former cases, to $\frac{v}{6}[2-r(3a_x-a_{xyz})+3a_{xy}-3a_{xz}]+\frac{1}{6l_x}[(v(1-3a_{x1\cdot y}+3a_{x1\cdot z}+a_{x1\cdot yz})\times l_{x1})-a_{1x\ yz}\,l_{1x}]+\frac{1}{6l_y}[v(1+a_{x\cdot y1\cdot z})l_{y1}+a_{x\cdot 1y\cdot z}\,2l_{1y}]-\frac{1}{6l_z}[v(1+a_{xy\cdot z1})\times 2l_{z1}+a_{xy\cdot 1z}\,l_{1z}]$, and which I shall denote by **G**. Therefore, when A is the oldest life, the required value will be represented by $s\times\mathbf{G}$.

§ 306. *Case* 2. Let B be the oldest of the three lives. In this case the first m terms of the several collateral series above mentioned will be equal to $\mathbf{G}-\frac{v(1-ra_{x+m})}{2}\cdot\frac{l_{x+m}\,v^m}{l_x}+\frac{v(1+a_{xz|m})}{2}\cdot\frac{l_x\,l_{z|m}\,v^m}{l_x\,l_z}-\frac{v(1+a_{x1\cdot z|m})}{2}\cdot\frac{l_{x1}\,l_{z|m}\,v^m}{l_x\,l_z}$. But, after the decease of B, the expectations for all the subsequent years may be more correctly expressed by means of the first Lemma in the fifth Chapter; for, since the chance of receiving the sum at the end of any one of those years depends on A's failing in the year, C having died before B in either of the preceding years (the probability of which is by the Lemma denoted by $1-f$), we shall find that the sum of the expectations for those years, continued to the utmost extent of human life, will be equal to $s(1-f)\,v(1-a_{x+m})\frac{l_{x+m}\,v^m}{l_x}$. Consequently this value, added to the sum of the first m terms of the series above found, will express the total present value of the given sum in this case required, and which will be found equal to $s[\mathbf{G}+k]$.

§ 307. *Case* 3. Let C be the oldest of the three lives. In this case the first m terms of the several collateral series above mentioned will be equal to $\mathbf{G}-\frac{v(1-ra_{x+m})}{2}\cdot\frac{l_{x+m}\,v^m}{l_x}-\frac{v(1+a_{xy|m})}{2}\cdot\frac{l_x\,l_{y|m}\,v^m}{l_x\,l_y}+\frac{v(1+a_{x1\cdot y|m})}{2}\cdot\frac{l_{x1}\,l_{y|m}\,v^m}{l_x\,l_y}$. But, after the decease of C, the chance of receiving the given sum at the end of any subsequent year will depend solely on the death of A, C having died before B in either of the preceding years; whence the sum of the expectations for all the subsequent years will, as in the preceding case, be denoted by $s(1-f)\,v(1-a_{x+m})\frac{l_{x+m}\,v^m}{l_x}$. Consequently this value, added to the sum of the first m terms of the series above found, will represent the total present value of the given sum in this case required, and which will be found equal to $s[G+u]$.[1]

COROLLARY.

§ 308. When the three lives are equal, or all of the same age A, the last three quantities in the formula denoted by G destroy each other, and the general expression will in this case become $s\times\frac{v}{6}[2-r(3a_x-a_{xxx})]$.

PROBLEM XLV.[2]

§ 309. To determine the present value of a given sum payable on the decease of A, provided he be the *first* or *last* that fails of three given lives, A, B, C, and provided C, in the latter case, dies before B.

SOLUTION.

The given sum may be received at the end of the first year on the happening of either of four different events: 1. that all the lives fail in that year, A having died first, or C having died first, B second, and A last; 2. that A and B fail in that year, A having died first, and that C lives to the end of it; 3. that A and C fail in that year, A having died first, and that B lives to the end of it; 4. that only A dies in that year, and that B and C both live to the end of it. The probabilities of the happening of these several events are respectively $\frac{d_x d_y d_z}{2l_x l_y l_z}$, $\frac{d_x d_y l_{z1}}{2l_x l_y l_z}$, $\frac{d_x d_z l_{y1}}{2l_x l_y l_z}$, and $\frac{d_x l_{y1} l_{z1}}{l_x l_y l_z}$: which being added together and multiplied by sv will give the expectation of receiving the sum at the end of the first year.

But, in the second and following years, the given sum may be received on the happening of either of six different events: 1. that all the lives fail in the year, A having died first, or C having died first, B second, and A last; 2. that A and B fail in the year, A having died first, and that C lives to the end of it; 3. that A and C fail in the year, A having died first, and that B lives to the end of it; 4. that only A dies in the year, and that B and C both live; 5. that A and B both fail in the year, A having died last, and C having failed in either of the preceding years; 6. that only A fails in the year, and *that B and C both fail in either of the*

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[1] It may be here useful to remark that the present value found by this Problem is equal to the sum of the two values found by Probs. XLI. and XLII.: and their agreement in this particular is a proof of the accuracy of the investigation.

[2] *Phil. Trans.* for 1800, Prob. 6. p. 37.

preceding years, C having died first. The probabilities of the happening of these several events in the second year are respectively $\frac{d_x\, d_y\, d_{z1}}{2l_x\, l_y\, l_z}$, $\frac{d_{x1}\, d_{y1}\, l_{z2}}{2l_x\, l_y\, l_z}$, $\frac{d_{x1}\, d_{z1}\, l_{y2}}{2l_x\, l_y\, l_z}$, $\frac{d_{x1}\, l_{y2}\, l_{z2}}{l_x\, l_y\, l_z}$, $\frac{d_{x1}\, d_{y1}}{2l_x\, l_y}(1-\frac{l_{z1}}{l_z})$, and $\frac{d_{x1}}{2l_x}(1-\frac{l_{y1}}{l_y})\times(1-\frac{l_{z1}}{l_z})$: which being added together and multiplied by sv^2 will give the expectation of receiving the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third and every subsequent year during the continuance of the oldest of the given lives: and if these several annual expectations be reduced to their lowest terms, and arranged under each other, they will form twelve collateral series; the first m terms of which will vary according to the seniority of the lives concerned.

§ 310. *Case* 1. Let A be the oldest of the three lives. In this case the several collateral series here mentioned must be continued to the utmost extent of human life, because the life A is involved in every term of it: therefore the sum of it will be equal to s multiplied into $\frac{v(1-ra_x)}{2}+\frac{a_{xy}}{2}-a_{1x\cdot y}\frac{l_{1x}}{2l_x}-\frac{v(1+a_{xz})}{2}+\frac{v(1+a_{x1\cdot z})}{2}\cdot\frac{l_{x1}}{l_x}+\frac{v(1-ra_{xyz})}{2}-\frac{v(1+a_{x1\cdot yz})}{2}\frac{l_{x1}}{l_x}+a_{1x\cdot yz}\frac{l_{1x}}{2l_x}+\frac{v(1+a_{x\cdot y1\cdot z})}{2}\cdot\frac{l_{y1}}{l_y}-a_{xy\cdot 1z}\frac{l_{1z}}{l_z}$: an expression which, independent of the common multiple s, may be reduced to $\frac{v}{2}[1-r(a_x+a_{xyz})-a_{xz}]+\frac{a_{xy}}{2}+\frac{1}{2l_x}[v(a_{x1\cdot z}-a_{x1\cdot yz})\,l_{x1}-(a_{1x\cdot y}-a_{1x\cdot yz})\,l_{1x}]+\frac{v}{2}(1+a_{x\cdot y1\cdot z})\frac{l_{y1}}{l_y}-a_{xy\cdot 1z}\frac{l_{1z}}{2l_z}$, and which I shall denote by **H**. Therefore, when A is the oldest of the three lives, the value required will be equal to $s\times\mathbf{H}$.[1]

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[1] It is really amusing to observe the manner in which Mr. Morgan has summed up the several series arising from this problem; and since this manner is by no means unusual with him, and as it gives me an opportunity of explaining more fully what I have already said in the preceding pages on his singular and confused method of investigating such cases, I shall here (for the last time) enter once more on the subject.

He makes the total value of the series, continued to the utmost extent of human life, equal to S into $\frac{r-1}{2r}(\mathrm{V}-\mathrm{A}-\mathrm{ABC})+\frac{\mathrm{AC}}{2}-\frac{\mathrm{AB}}{2r}+\frac{md}{2bcr}(1+\mathrm{APT})-\frac{d}{2cr}(1+\mathrm{AT}+\mathrm{BT}-\mathrm{ABT})-\frac{\beta\kappa}{2bc}\times\mathrm{AFK}+\frac{\beta}{2b}(\mathrm{AF}+\mathrm{FC}-\mathrm{AFC})$: See *Phil. Trans.* for 1800, p. 37. Now, this is certainly the correct value of the series; but it is necessary to remark that $\frac{\beta.\mathrm{FC}}{2b}$ is

§ 311. *Case* 2. Let B be the oldest of the three lives. In this case the sum of the first m terms of the several collateral series above mentioned will be equal to $\mathbf{H}-\frac{v(1-ra_{x+m})}{2}\frac{l_{x+m}\,v^m}{l_x}+\frac{v(1+a_{xz|m})}{2}\cdot\frac{l_x\,l_{z|m}\,v^m}{l_x\,l_z}-\frac{v(1+a_{x1\cdot z|m})}{2}\cdot\frac{l_{x1\cdot z|m}\,v^m}{l_x\,l_z}$. But, after the decease of B, the value of the expectations for all the subsequent years may be more correctly expressed, as in the second case to Prob. XL., by $s(1-f)\,v(1-ra_{x+m})\cdot\frac{l_{x+m}\,v^m}{l_x}$: and which being added to the sum of the first m terms found as above, will make the total present value of the given sum, when B is the oldest of the three lives, equal to $v(\mathbf{H}+k)$.

§ 312. *Case* 3. Let C be the oldest of the three lives. In this case the first m terms of the several collateral series above mentioned will be equal to $\mathbf{H}-\frac{v(1-ra_{x+m})}{2}\cdot\frac{l_{x+m}\,v^m}{l_x}-a_{xy|m}\frac{l_x\,l_{y|m}\,v^m}{2l_x\,l_y}+a_{1x\cdot y|m}\times\frac{l_{1x\cdot y|m}\,v^m}{2l_x\,l_y}$. But, after the decease of C, the chance of receiving the given sum at the end of any subsequent year will depend on the happening of either of two events: 1. that A dies after B in the year; 2. that only A dies in the year, B having died after C in either of the preceding years. The sum

equal to $\frac{d}{2cr}(1+\mathrm{BT})$; and, as these two quantities are used with *contrary signs*, it is evident that they ought not to have been introduced into the equation. Consequently the formula should have been reduced to $\frac{r-1}{2r}\times(\mathrm{V}-\mathrm{A}-\mathrm{ABC})+\frac{\mathrm{AC}}{2}-\frac{\mathrm{AB}}{2r}+\frac{md}{2bcr}(1+\mathrm{APT})-\frac{d}{2cr}\times(\mathrm{AT}-\mathrm{ABT})-\frac{\beta\kappa}{2bc}\times\mathrm{AFK}+\frac{\beta}{2b}(\mathrm{AF}-\mathrm{AFC})$.

The reader will observe that this is the total value of the series, continued to the utmost extent of human life; and, therefore, that it will truly denote the value of the given sum *when* A is the *eldest of the three lives.* Mr. Morgan, however, when he comes to consider that case, says that the symbols *must be changed;* and he then makes the given sum equal to S into $\frac{r-1}{2r}(\mathrm{V}-\mathrm{A}-\mathrm{ABC})+\frac{\mathrm{AC}}{2}-\frac{\mathrm{AB}}{2r}-\frac{\kappa.\mathrm{AK}}{2c}+\frac{\alpha\mathrm{HBC}}{2a}-\frac{d}{2cr}(\mathrm{AT}-\mathrm{ABT})+\frac{s}{2ar}(1+\mathrm{NC}+\mathrm{NB}-\mathrm{NBC})-\frac{ds}{2acr}(1+\mathrm{NBT})$. Now, a moment's attention to this formula will show us that it does not in reality differ from the preceding one: for $\frac{md}{2bcr}(1+\mathrm{APT})=\frac{\alpha\mathrm{HBC}}{2a}$; $\frac{\beta}{2b}(\mathrm{AF}-\frac{\kappa}{c}\mathrm{AFK})=\frac{s}{2ar}(\mathrm{NB}-\mathrm{NBC})$; $\frac{\beta.\mathrm{AFC}}{2b}=\frac{ds}{2acr}(1+\mathrm{NBT})$; whence these values may be safely substituted for each other, since they by no means denote *different* quantities, but arise merely from the different methods of summing up the same series, agreeably to what I have before remarked. But, as the first formula had been already deduced by him, and was stated at length, it certainly did not become necessary to confuse the subject by introducing a second. The most remarkable circumstance, however, attending this investigation is that (after discarding the two useless quantities $\frac{\beta.\mathrm{FC}}{2b}-$

of the expectations on the happening of these events has been already found, by the 3d case in Prob. XLII., to be equal to $(1-f)\,v(1-ra_{x+m})\times\frac{l_{x+m}\,v^m}{l_x}-\frac{v\,(1+a_{xy|m}}{2}\cdot\frac{l_x\,l_{y|m}\,v_m}{l_x\,l_y}+a_{xy|m}\frac{l_x\,l_{y|m}\,v^m}{2l_x\,l_y}+\frac{v\,(1+a_{x1\cdot y|m}}{2}\times\frac{l_{x1}\,l_{y|m}\,v^m}{l_x\,l_y}-a_{1x\cdot y|m}\frac{l_{1x\cdot y|m}\,v^m}{2l_x\,l_y}$: which being added to the sum of the first m terms of the several collateral series above mentioned, will make the total present value required equal to $s(\mathbf{H}+u)$.[1]

COROLLARY.

§ 313. When the three lives are equal, or all of the same age A, the last three quantities in the formula denoted by **H** destroy each other, and the general expression in this case will become equal to $\frac{sv}{2}[1-r(a_x-a_{xx}+a_{xxx})]$.

PROBLEM XLVI.[2]

§ 314. To determine the present value of a given sum payable on the decease of A and B, provided they be the *first* that fail of three given lives, A, B, C.

$\frac{d}{2cr}(1+\mathrm{BT})$ from this second formula) he should insert in their stead two *other* useless quantities, $\frac{\kappa.\mathrm{AK}}{2c}$ and $\frac{s}{2ar}(1+\mathrm{NC})$: which being equal to each other, and used with *contrary signs*, ought of course to have been discarded also. Surely Mr. Morgan could not have been writing either for the *convenience* or the *information* of mankind ! ! !

I shall here take the opportunity of repeating the observation made in the note to § 253; that, although the quantities here alluded to are strictly and mathematically *equal*, and being used with contrary signs *destroy each other*, yet (owing to the imperfection of the Tables which show the value of Life Annuities) they oftentimes produce a *real* expression when solved arithmetically. Thus $\frac{s}{ar}(1+\mathrm{NC})-\frac{\kappa.\mathrm{AK}}{c}$, which, expressed by the characters made use of in this work, becomes $v(1+a_{x1\cdot z})\frac{l_{x1}}{l_x}-a_{x\cdot 1z}\frac{l_{1z}}{l_z}$, is, from what has been said in the note to § 231, equal to 0: but if reduced to *numbers* (taking the age of A 15 years, the age of C 18 years, the rate of interest 4 per cent., and the probabilities of living as at *Northampton*) it will become equal to 13·362 — 13·348 = ·014. This fact will sufficiently show how necessary it is to discard all *superfluous* quantities; and that Mr. Morgan's formulæ are, at best, but approximations.

[1] It may be useful to remark that the present value found by this problem is equal to the sum of the two values found by Prob. XXIX. and XLII., which confirms the accuracy of the investigation.

[2] Simpson's *Sup.* Prob. 39. Dodson, vol. iii. Ques. 48. Morgan, Prob. 22; and in *Phil. Trans.* for 1791, Prob. 4, p. 258.

SOLUTION.

In order to receive the sum at the end of the first year, either of these two events must happen: 1. that all the lives fail in that year, C having died last; 2. that the two lives A and B fail in that year, and that C lives to the end of it. The probabilities of the happening of these two events are respectively $\frac{d_x\, d_y\, d_z}{3l_x\, l_y\, l_z}$, and $\frac{d_x\, d_y\, l_{z1}}{l_x\, l_y\, l_z}$: which being added together and multiplied by *sv* will give the expectation of receiving the sum at the end of the first year.

But, in the second and following years, the given sum may be received on the happening of either of six different events: 1. that all the lives fail in the year, C having died last; 2. that A and B both fail in the year, and that C lives to the end of it; 3. that A and C both fail in the year, A having died first, and that B dies in either of the preceding years; 4. that B and C both fail in the year, B having died first, and that A dies in either of the preceding years; 5. that only A dies in the year, C living to the end of it, and B having died in either of the preceding years; 6. that only B dies in the year, C living to the end of it, and A having died in either of the preceding years. The probabilities of the happening of these several events in the second year are respectively $\frac{d_{x1}\, d_{y1}\, d_{z1}}{3l_x\, l_y\, l_z}$, $\frac{d_{x1}\, d_{y1}\, l_{z2}}{l_x\, l_y\, l_z}$, $\frac{d_{x1}\, d_{z1}}{2l_x\, l_z}(1-\frac{l_{y1}}{l_y})$, $\frac{d_{y1}\, d_{z1}}{2l_y\, l_z}(1-\frac{l_{x1}}{l_x})$, $\frac{d_{x1}\, l_{z2}}{l_x\, l_z}(1-\frac{l_{y1}}{l_y})$, and $\frac{d_{y1}\, l_{z2}}{l_y\, l_z}(1-\frac{l_{x1}}{l_x})$: which being added together and multiplied by *sv* will give the expectation of receiving the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third and every subsequent year, to the utmost extent of human life: and if these several expectations be reduced to their lowest terms, and arranged under each other, they will form sixteen collateral series, the sum of which will be found equal to $s[A_C+B_C-AB_C]$.

COROLLARY.

§ 315. When the lives are all equal to each other, or of the same age A, this expression (agreeably to what has been said in the corollary to Prob. XXX.) will become equal to $\frac{sv}{3}[1-r(3a_{xx}-2a_{xxx})]$; that is, equal to one-third of the present value of the given sum payable on the extinction of any two out of the three given lives.

PROBLEM XLVII.[1]

§ 316. To determine the present value of a given sum payable on the decease of A *and* B, provided they be the *last* that fail of the three given lives, A, B, C.

SOLUTION.

It is evident that the given sum cannot be received at the end of the first year unless all the lives become extinct, C having died first; the probability of which is $\frac{d_x\, d_y\, d_z}{3l_x\, l_y\, l_z}$; and which being multiplied by sv will give the expectation of receiving the sum at the end of the first year.

But, in the second and following year, the given sum may be received on the happening of either of four different events: 1. that all the lives fail in the year, C having died first; 2. that A and B fail in the year, C having died in either of the preceding years; 3. that only A fails in the year, and that B *and* C *both fail in either of the preceding years,* B *having died last;* 4. that only B fails in the year, and *that* A *and* C *both fail in either of the preceding years,* A *having died last.* The probabilities of the happening of these several events in the second year are respectively, $\frac{d_x\, d_y\, d_{z|1}}{3l_x\, l_y\, l_z}$, $\frac{d_{x1}\, d_{y1}}{l_x\, l_y}(1-\frac{l_{z1}}{l_z})$, $\frac{d_{x1}}{2l_x}(1-\frac{l_{y1}}{l_y})(1-\frac{l_{z1}}{l_z})$, and $\frac{d_{y1}}{2l_y}(1-\frac{l_{x1}}{l_x})(1-\frac{l_{z1}}{l_z})$: which being added together and multiplied by sv^2 will give the expectation of receiving the sum at the end of the second year. In like manner, we may find the expectation of receiving the sum at the end of the third and every subsequent year: and if these several expectations be reduced to their lowest terms and arranged under each other, they will form nineteen collateral series, the sum of which, if continued to the utmost extent of human life, would be equal to s multiplied into $\frac{v(1-ra_x)}{2}+\frac{v(1-ra_y)}{2}+a_{xy}-\frac{v(1+a_{x1\cdot y})}{2}\cdot\frac{l_{x1}}{l_x}-\frac{v(1+a_{x\cdot y1})}{2}\cdot\frac{l_{y1}}{l_y}-\frac{v(1+a_{xz})}{2}+\frac{v(1+a_{x1\cdot z})}{2}\cdot\frac{l_{x1}}{l_x}-\frac{v(1+a_{yz})}{2}+\frac{v(1+a_{y1\cdot z})}{2}\cdot\frac{l_{y1}}{l_y}+\frac{v(1-ra_{xyz})}{3}+\frac{v(1+a_{x1\cdot yz})}{6}\cdot\frac{l_{x1}}{l_x}+\frac{{}_{1x\cdot yz}\,l_{1z}}{3l_x}+\frac{v(1+a_{x\cdot y1\cdot z})}{6}\cdot\frac{l_{y1}}{l_y}+\frac{a_{x\cdot 1y\cdot z}\,l_{1y}}{3l_y}-\frac{v(1+a_{xy\cdot z1})}{3}\cdot\frac{l_{z1}}{l_z}-\frac{a_{xy\cdot 1z}\,2l_{1z}}{3l_z}$; an expression which, independent of the common multiple s, may be reduced to $\frac{v}{6}[2-r(3a_x+3a_y+2a_{xyz})-3a_{xz}-3a_{yz}]+a_{xy}+\frac{1}{6l_x}[v(1-3a_{xy}+3a_{x1\cdot z}+$

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[1] Morgan, Prob. 23; and in *Phil. Trans.* for 1794, Prob. 5, p. 247.

$a_{x1\cdot yz}) \, l_{x1} + a_{1x\cdot yz} \, 2l_{1x}] + \frac{1}{6l_y}[v(1-3a_{x\cdot y1}+3a_{y1\cdot z}+a_{x\cdot y1\cdot z})l_{y1}+a_{x\cdot 1y\cdot z}2l_{1y}]$ $-\frac{1}{3l_z}[v(1+a_{xy\cdot z1}) \,.\, l_{z1}+a_{xy\cdot 1z}2l_{1z}]$; and which I shall denote by J.

§ 317. *Case* 1. Let A be the oldest of the three lives. In this case the sum of the first m terms of the several collateral series above mentioned will be equal to $\mathrm{J}-\frac{v(1-ra_{y+m})}{2} \,.\, \frac{l_{y+m}\,v^m}{l_y}+\frac{v(1+a_{yz|m})}{2} \,.\, \frac{l_y\,l_{z|m}\,v^m}{l_y\,l_z}-\frac{v(1+a_{y1\cdot z|m})}{2} \,.\, \frac{l_{y1\cdot z|m}\,v^m}{l_y\,l_z}$. But, after the decease of A, the sum of the expectations for all the subsequent years will, as in the first case of Prob. XL., be more correctly expressed by $s(1-q)\,v(1-ra_{y+m})\frac{l_{y+m}\,v^m}{l_y}$; and which being added to the first m terms just found will make the total present value, when A is the oldest life, equal to $s(\mathrm{J}+h)$.

§ 318. *Case* 2. Let B be the oldest of the three lives. In this case the sum of the first m terms of the several collateral series above mentioned will be equal to $\mathrm{J}-\frac{v(1-ra_{x+m})}{2} \,.\, \frac{l_{x+m}\,v^m}{l_x}+\frac{v(1+a_{xz|m})}{2} \,.\, \frac{l_x\,l_{z|m}\,v^m}{l_x\,l_z}-\frac{v(1+a_{x1\cdot z|m})}{2} \,.\, \frac{l_{x1\cdot z|m}\,v^m}{l_x\,l_z}$. But, after the decease of B, the sum of the expectations for all the subsequent years will, as in the second case of Prob. XL., be more correctly expressed by $s(1-f)\,v(1-ra_{x+m})\frac{l_{x+m}\,v^m}{l_x}$; and which being added to the first m terms just found, will make the total present value, when B is the oldest life, equal to $s(\mathrm{J}+k)$.

§ 319. *Case* 3. Let C be the oldest of the three lives. In this case the sum of the first m terms of the several collateral series above mentioned will be equal to $\mathrm{J}-\frac{v(1-ra_{x+m})}{2} \,.\, \frac{l_{x+m}v^m}{l_x}-\frac{v(1-ra_{y+m})}{2} \,.\, \frac{l_{y+m}\,v^m}{l_y}-a_{xy|m}\times \frac{l_x\,l_{y|m}v^m}{l_x\,l_y}+\frac{v(1+a_{x1\cdot y|m})}{2} \,.\, \frac{l_{x1\cdot y|m}\,v^m}{l_x\,l_y}+\frac{v(1+a_{x\cdot y1|m})}{2} \,.\, \frac{l_x l_{y1|m}v^m}{l_x\,l_y}$. But, after the decease of C, the sum of the expectations for all the subsequent years will, as in the third case of Prob. XL., be more correctly expressed by s multiplied into $(1-f)\,v(1-ra_{x+m}) \,.\, \frac{l_{x+m}\,v^m}{l_x}+(1-q)v(1-ra_y)\frac{l_{y+m}\,v^m}{l_y}-v(1+a_{xy|m}) \,.\, \frac{l_x\,l_{y|m}v^m}{l_x\,l_y}+a_{xy|m}\,\frac{l_x\,l_{y|m}v^m}{l_x\,l_y}$; and which being added to the first m terms just found, will make the total present value, when C is the oldest life, equal to $s(\mathrm{J}+p+u)$.

COROLLARY.

§ 320. When all the lives are equal, or of the same age A, the last three quantities in the formula denoted by J destroy each other, and the general expression in this case will become equal to $\frac{sv}{3}[1-r(3a_x-3a_{xx}+a_{xxx})]$: that is, equal to one-third of the present value of the given sum to be received on the extinction of the longest of the three lives.

PROBLEM XLVIII.[1]

§ 321. To determine the present value of a given sum payable on the decease of A and B, provided they be the *first* and *last* that fail of three given lives, A, B, C.

SOLUTION.

The given sum cannot be received at the end of the first year, unless all the three lives are extinct, C having died second; the probability of which is $\frac{d_x d_y d_z}{3l_x l_y l_z}$; and which being multiplied by *sv* will give the expectation of receiving the sum at the end of the first year.

But, in the second and following years, the given sum may be received on the happening of either of five different events: 1. that all the lives fail in the year, C having died second; 2. that A and C both fail in the year, A having died last, and B having failed in either of the preceding years; 3. that B and C both fail in the year, B having died last, and A having failed in either of the preceding years; 4. that only A dies in the year, and that B and C both fail in either of the preceding years, B having died first; 5. that only B dies in the year, and that A and C both fail in either of the preceding years, A having died first. The probabilities of the happening of these several events in the second year are respectively $\frac{d_x d_y d_{z|1}}{3l_x l_y l_z}$, $\frac{d_{x1} d_{z1}}{2l_x l_z}(1-\frac{l_{y1}}{l_y})$, $\frac{d_{y1} d_{z1}}{2l_y l_z}(1-\frac{l_{x1}}{l_x})$, $\frac{d_{x1}}{2l_x}(1-\frac{l_{y1}}{l_y})$, $(1-\frac{l_{z1}}{l_z})$, and $\frac{d_{y1}}{2l_y}(1-\frac{l_{x1}}{l_x})(1-\frac{l_{z1}}{l_z})$: which being added together and multiplied by sv^2 will give the expectation of receiving the sum at the end of the second year. In like manner we may find the expectation of receiving the sum at the end of the third and every subsequent year: and if these several annual expectations be reduced to their lowest terms, and arranged under each other, they will form nineteen collateral series; the sum of which, if continued to the utmost extent of human life, would be equal to s multiplied into $\frac{v(1-ra_x)}{2}+\frac{v(1-ra_y)}{2}-v(1+a_{xy})+\frac{v(1+a_{x1\cdot y})}{2}\cdot\frac{l_{x1}}{l_x}+\frac{v(1+a_{x\cdot y1})}{2}\cdot\frac{l_{y1}}{l_y}+\frac{a_{xz}}{2}-a_{1x\cdot z}\frac{l_{1x}}{2l_x}+\frac{a_{yz}}{2}-a_{1y\cdot z}\frac{l_{1y}}{2l_y}+\frac{v(1-ra_{xyz})}{3}-\frac{v(1+a_{x1\cdot yz})}{3}\cdot\frac{l_{x1}}{l_x}-a_{1x\cdot yz}\frac{l_{1x}}{6l_x}-\frac{v(1+a_{x\cdot y1\cdot z})}{3}\frac{l_{y1}}{l_y}-a_{x\cdot 1y\cdot z}\frac{l_{1y}}{6l_y}+\frac{v(1+a_{xy\cdot z1})}{3}\frac{2l_{z1}}{l_z}+a_{xy\cdot 1z}\frac{l_{1z}}{3l_z}$. But this expression, independent of the common multiple s, may be reduced to $\frac{v}{6}[2-r(3a_x+3a_y+2a_{xyz})-6a_{xy}]+\frac{a_{xz}+a_{yz}}{2}+\frac{1}{6l_x}[v(1+3a_{x1\cdot y}-2a_{x1\cdot yz})\,l_{x1}-(3a_{1x\cdot z}+a_{1x\cdot yz})l_{1x}]+\frac{1}{6l_y}[v(1+3a_{x\cdot y1}-2a_{x\cdot y1\cdot z})\,l_{y1}-(3a_{1y\cdot z}+a_{x\cdot 1y\cdot z})\,l_{1y}]+\frac{1}{3l_z}[v(1+a_{xy\cdot z1})\,2l_{z1}+a_{xy\cdot 1z}]$; and I shall denote it by **K**.

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[1] This case appears to have been omitted by Mr. Morgan; and no solution of it, prior to the present one, has hitherto been published.

§ 322. *Case* 1. Let A be the oldest of the three lives. In this case the first m terms of the several collateral series here alluded to will be equal to $\mathbf{K}-\frac{v(1-ra_{y+m})}{2}\cdot\frac{l_{y+m}\,v^m}{l_y}-a_{yz|m}\frac{l_y\,l_{z|m}\,v^m}{2l_y\,l_z}+a_{y\cdot 1z|m}\frac{l_{1y}\,l_{z|m}\,v^m}{2l_y\,l_z}$. But, after the decease of A, the chance of receiving the sum at the end of any subsequent year will depend on the happening of two different events: 1. that B and C both fail in the year, B having died last; 2. that only B dies in the year, C having died after A in either of the preceding years. The probabilities of the happening of these two events in the $(m+1)$st year are respectively $\frac{d_y\,d_{z|m}}{2l_y\,l_z}$, and $\frac{d_{y+m}}{l_y}(q-\frac{l_{z+m}}{l_z})$:[1] which being added together and multiplied by sv^{m+1} will give the expectation of receiving the sum at the end of the $(m+1)$st year. In like manner the probabilities of the happening of these two events in the $(m+2)$nd year are respectively $\frac{d_y d_{z|m}}{2l_y\,l_z}$, and $\frac{d_{y+m}}{l_y}(q-\frac{l_{z1+m}}{l_z})$: which being added together and multiplied by sv^{m+2} will give the expectation of receiving the sum at the end of the $(m+2)$nd year; and so on for all the subsequent years to the utmost extremity of human life. But the sum of all these annual expectations will be found equal to $qv(1-ra_{y+m})\frac{l_{y+m}\,v^m}{l_y}-\frac{v(1+a_{yz|m})}{2}\cdot\frac{l_y\,l_{z|m}\,v^m}{l_y\,l_z}+a_{yz|m}\times\frac{l_y\,l_{z|m}\,v^m}{2l_y\,l_z}+\frac{v(1+a_{y1\cdot z|m})}{2}\cdot\frac{l_{y1\cdot z|m}\,v^m}{l_y\,l_z}-a_{1y\cdot z}\frac{l_{1y}\,l_z\,v^m}{2l_y\,l_z}$; and which being added to the first m terms of the several collateral series above mentioned will make the total present value of the given sum, when A is the oldest life, equal to $s(\mathbf{K}-h)$.

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[1] See the Scholium in § 156.

§ 323. *Case* 2. Let B be the oldest of the three lives. In this case the first m terms of the several collateral series above mentioned will be equal to $\mathbf{K} - \frac{v(1-ra_{x+m})}{2} \cdot \frac{l_{x+m}\, v^m}{l_x} - a_{xz|m} \frac{l_x\, l_{z|m}\, v^m}{2l_x\, l_z} + a_{1x\cdot z|m} \frac{l_{1x} l_{z|m} v^m}{2l_x\, l_z}$. But, after the decease of B, the chance of receiving the sum at the end of any subsequent year will depend on the happening of two different events: 1. that A and C both fail in the year, A having died last; 2. that only A dies in the year, C having died after B in either of the preceding years. The probabilities of the happening of these two events in the $(m+1)$st year, are respectively $\frac{d_x\, d_{z|m}}{2l_x\, l_z}$, and $\frac{d_{x+m}}{l_x}(f - \frac{l_{z+m}}{l_z})$: which being added together and multiplied by sv^{m+1} will give the expectation of receiving the sum at the end of the $(m+1)$st year.

In like manner, we might find the expectation of receiving the sum at the end of the $(m+2)$nd and every subsequent year, to the utmost extremity of human life; the sum of all which annual expectations will be found equal to s multiplied into $f - v(1-ra_{x+m}) \frac{l_{x+m}\, v^m}{l_x} - \frac{v(1+a_{xz|m})}{2} \cdot \frac{l_x\, l_{z|m}\, v^m}{l_x\, l_z} + a_{xz|m} \frac{l_x\, l_{z|m}\, v^m}{2l_x\, l_z} + \frac{v(1+a_{x1\cdot z|m})}{2} \cdot \frac{l_{x1\cdot z|m}\, v^m}{l_x\, l_z}$ $- a_{1x\cdot z|m} \frac{l_{1x\cdot z|m}\, v^m}{2l_x\, l_z}$; and which being added to the first m terms of the several collateral series above mentioned will make the present value of the given sum, when B is the oldest of the three lives, equal to $s(\mathbf{K} - k)$.

§ 324. *Case* 3. Let C be the oldest of the three lives. In this case the sum of the first m terms of the several collateral series above mentioned will be equal to $\mathbf{K} - \frac{v(1-ra_{x+m})}{2} \cdot \frac{l_{x+m}\, v^m}{l_x} - \frac{v(1-ra_{y+m})}{2} \cdot \frac{l_{y+m}\, v^m}{l_y}$ $+ \frac{v(1+a_{xy|m})\, l_x\, l_{y|m}\, v^m}{l_x\, l_y} - \frac{v(1+a_{x1\cdot y|m})}{2} \cdot \frac{l_{x1\cdot y|m}\, v^m}{l_x\, l_y} - \frac{v(1+a_{x\cdot y1|m})}{2} \cdot$ $\frac{l_x\, l_{y1|m}\, v^m}{l_x\, l_y}$. But, after the decease of C, the chance of receiving the sum at the end of any subsequent year will depend on the happening of two different events: 1. that only A dies in the year, C having died after B in either of the preceding years (the probability of which latter contingency will now be denoted by f); 2. that only B dies in the year, C having died after either of the preceding years (the probability of which latter contingency will now be denoted by q). Consequently, the sum of the expectations on the happening of these events, continued to the utmost extent of human life, will be equal to $sfv(1-ra_{x+m}) \frac{l_{x+m}\, v^m}{l_x} + sqv(1-ra_{y+m}) \frac{l_{y+m}\, v^m}{l_y}$; and which being added to the sum of the first m terms of the several collateral series above mentioned will make the total present value of the given sum, when C is the oldest life, equal to $s(\mathbf{K}-p-u)$.[1]

COROLLARY.

§ 325. When the lives are equal, or all of the same age A, the last three quantities in the formula denoted by **K** destroy each other, and the general expression in this case will become $\frac{sv}{3}[1-r(a_x-3a_{xx}+a_{xxx})]$: that is, equal to one-third of the present value of the given sum to be received on the extinction of the longest of the three lives.

PROBLEM XLIX.[2]

§ 326. To determine the present value of a given sum payable on the decease of A and B, provided B dies before another life C.

SOLUTION.

The payment of the sum at the end of the first year will depend on the happening of either of two events: 1. that all the lives fail in the year, B having died before C; 2. that only A and B die in that year, C living to the end of it. The probabilities of the happening of these two events are respectively $\frac{d_x d_y d_z}{2l_x l_y l_z}$, and $\frac{d_x d_y l_{z1}}{l_x l_y l_z}$: which being added together and multiplied by *sv* will give the expectation of receiving the sum at the end of the first year.

But, in the second and following years, the given sum may be received on the happening of either of seven different events: 1. that all the lives fail in the year, B having died before C; 2. that A and B fail in the year, and that C lives to the end of it; 3. that A and C both fail in the year, and that B dies in either of the preceding years; 4. that B and C both fail in the year, B having died first, and that A dies in either of the preceding years; 5. that only A dies in the year, B having failed in either of the preceding years, and C living to the end of it; 6. that only B dies in the year, C living to the end of it, and A having died in either of the preceding years; 7. *that only* A *dies in the year*, *and that* B *and* C *both fail in either of the preceding years*, B *having died first*. The probabilities of the happening of these several events in the same year are respectively $\frac{d_{x1} d_{y1} d_{z1}}{2l_x l_y l_z}$, $\frac{d_{x1} d_{y1} l_{z1}}{l_x l_y l_z}$, $\frac{d_{x1} d_{z1}}{l_x l_z}(1-\frac{l_{y1}}{l_y})$, $\frac{d_{y1} d_{z1}}{2l_y l_z}(1-\frac{l_{x1}}{l_x})$, $\frac{d_{x1} l_{z2}}{l_x l_{z1}}(1-\frac{l_{y1}}{l_y})$,

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[1] It may not be improper here to remark that the present values of the sums, found by Problems XLVI. XLVII. and XVIII., are together equal to $sv[1-r(a_x+a_y-a_{xy})]$; or the present value of the given sum after the longest of the two lives A, B.

[2] Morgan, Prob. 28; and in *Phil. Trans.* for 1794, Prob. 3, p. 242.

$\frac{d_{y1}\,l_{z2}}{l_y\,l_z}(1-\frac{l_{x1}}{l_x})$, and $\frac{d_{x1}}{2l_x}(1-\frac{l_{y1}}{l_y})(1-\frac{l_{z1}}{l_z})$: which being added together and multiplied by sv^2 will give the expectation of receiving the sum at the end of the second year. In like manner may be found the expectation of receiving the sum at the end of the third and every subsequent year, during the continuance of the oldest life: and if these yearly expectations be reduced to their lowest terms and arranged under each other, they will form fourteen collateral series, the first m terms of which will vary according to the seniority of the lives concerned.

§ 327. *Case* 1. Let A be the oldest of the three lives. In this case, the first m terms of the several collateral series above mentioned will be equal to the sum of the terms continued to the utmost extent of human life; because the life of A is involved in each collateral series: therefore its value will be equal to s multiplied into $\frac{v(1-ra_x)}{2}-\frac{v(1+a_{xy})}{2}+\frac{v(1+a_{x1\cdot y})}{2}\cdot\frac{l_{x1}}{l_x}+\frac{v(1+a_{xz})}{2}-\frac{v(1+a_{x1\cdot z})}{2}\cdot\frac{l_{x1}}{l_y}+\frac{v(1-ra_{yz})}{2}-\frac{v(1+a_{y1\cdot z})}{2}\cdot\frac{l_{y1}}{l_y}+\frac{a_{1y\cdot z}\,l_{1y}}{2l_y}-\frac{v(1-ra_{xyz})}{2l_x}-\frac{a_{x\cdot 1y\cdot z}\,l_{1y}}{2l_y}+\frac{a_{xy\cdot 1z}\,l_{1z}}{2l_z}$. But this expression may be reduced to $\frac{v}{2}[1-r(a_x+a_{yz}-a_{xyz})-a_{xy}+a_{xz}]+\frac{v(a_{x1\cdot y}-a_{x1\cdot z})\,l_{x1}}{2l_x}-\frac{1}{2l_y}[v(1+a_{y1\cdot z})l_{y1}-(a_{1y\cdot z}-a_{x\cdot 1y\cdot z})l_{1y}]+\frac{a_{xy\cdot 1z}\,l_{1z}}{2l_z}$: and which I shall denote by **L**; consequently, when A is the oldest life, the present value required will be represented by s**L**.

§ 328. *Case* 2. Let B be the oldest of the three lives. In this case the first m terms of the several collateral series above mentioned will be equal to **L**$-\frac{v(1-ra_{x+m})}{2}\cdot\frac{l_{x+m}\,v^m}{l_x}-\frac{v(1+a_{xz|m})}{2}\cdot\frac{l_x\,l_{z|m}\,v^m}{l_x\,l_z}+\frac{v(1+a_{x1\cdot z|m})}{2}\cdot\frac{l_x\,l_{z|m}\,v^m}{l_x\,l_z}$. But, after the decease of B, the sum of the expectations for all the subsequent years, continued to the utmost extent of human life, will (as in the second case, Prob. XXXIX.) be equal to $sfv(1-ra_{x|m})\frac{l_{x|m}v^m}{l_x}$: which being added to the sum of the first m terms of the several collateral series above mentioned will make the total present value of the given sum equal to s(**L**$-k$).

§ 329. Let C be the oldest of the three lives. In this case the first m terms of the several collateral series above alluded to will be equal to **L**$-\frac{v(1-ra_{x|m})}{2}\cdot\frac{l_{x|m}\,v^m}{l_x}+\frac{v(1+a_{xy|m})}{2}\;\frac{l_x\,l_{y|m}\,v^m}{l_x\,l_y}-\frac{v(1+a_{x1\cdot y|m})}{2}\cdot\frac{l_{x1}\cdot l_{y|m}\,v^m}{l_x\,l_y}$.

But, after the decease of C, the sum of the expectations for all the subsequent years may be more correctly expressed, as in the preceding case, by $sfv(1-ra_{x|m})\frac{l_{x|m}\,v^m}{l_x}$; and which being added to the first m terms of the several collateral series above mentioned will make the total present value of the given sum, in this case required, equal to $s(\mathbf{L}-u)$.

COROLLARY.

§ 330. When the three lives are equal to each other, or of the same age A, the last three quantities in the formula denoted by **L** destroy each other, and the general expression will in this case become equal to $\frac{sv}{2}[1-r(a_x+a_{xx}-a_{xxx})]$.[1]

PROBLEM L.[2]

§ 331. To determine the present value of a given sum payable on the decease of A and B, provided another life C dies before B.

SOLUTION.

The payment of the given sum at the end of the first year will depend on the happening of one event only: viz., the extinction of all the lives in that year, C having died before B; the probability of this event is $\frac{d_x\,d_y\,d_z}{2l_x\,l_y\,l_z}$: which being multiplied by sv will give the expectation of receiving the sum at the end of the first year.

But in the second and following years, the given sum may be received on the happening of either of five different events: 1. that all the lives fail in the year, C having died before B; 2. that A and B fail in the year, and that C fails in either of the preceding years; 3. that B and C both fail in the year, C having died first, and that A dies in either of the preceding years; 4. that only B dies in the year, and that A and C both die in either of the preceding years; 5. that only A fails in the year, and *that* B *and* C *both fail in either of the preceding years*, B *having died last.* The probabilities of the happening of these several events in the second year will be respectively $\frac{d_x d_y d_{z|1}}{2l_x l_y l_z}$, $\frac{d_x d_{y|1}}{l_x l_y}(1-\frac{l_{z1}}{l_z})$, $\frac{d_y d_{z|1}}{2l_y l_z}(1-\frac{l_{x1}}{l_x})$, $\frac{d_{y1}}{l_y}(1-\frac{l_{x1}}{l_x})(1-\frac{l_{z1}}{l_z})$, and $\frac{d_{x1}}{2l_x}(1-\frac{l_{y1}}{l_y})(1-\frac{l_{z1}}{l_z})$: which being added together and multiplied by sv^2 will give the expectation of receiving the sum at the end of the second year. In like manner may be found the expectation of receiving the sum at the end of the third and every subsequent year during the continuance of the oldest life: and if these several expectations be reduced to their lowest terms and arranged under each other, they will form seventeen collateral series; the sum of the first m terms of which will vary according to the seniority of the lives concerned.

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[1] Mr. Morgan makes the value of the sum in this case equal to $\frac{s(r-1)}{r}\times(V-CC-C-CCC)$; that is, in the present notation, equal to $sv[1-r(a_x+a_{xx}\mp a_{xxx}]$: an error which has also been faithfully transcribed into two successive editions of Dr. Price's treatise. See *Phil. Trans.* for 1794, Part ii. p. 245; and Dr. Price's *Obs. on Rev. Pay.*, vol. i. p. 395.

[2] *Phil. Trans.* for 1794, Prob. 4, p. 245; and for 1800, Prob. 7, p. 40; which two *might* have been comprised in *one* problem.

§ 332. *Case* 1. Let A be the oldest of the three lives. In this case the several collateral series here mentioned must be continued to the utmost extent of human life; because the life of A is involved in each perpendicular series; whence, its value will be equal to s multiplied into $\frac{v(1-ra_x)}{2}+\frac{v(1-ra_y)}{2}-\frac{v(1+a_{xy})}{2}+a_{xy}-\frac{v(1+a_{x1\cdot y})}{2}\cdot\frac{l_{x1}}{l_x}-\frac{v(1+a_{xz})}{2}+\frac{v(1+a_{x1\cdot z})}{2}\cdot\frac{l_{x1}}{l_x}-\frac{v(1-ra_{yz})}{2}+\frac{v(1+a_{y1\cdot z})}{2}\cdot\frac{l_{y1}}{l_y}-\frac{a_{1y\cdot z}\,l_{1y}}{2l_y}+\frac{v(1-ra_{xyz})}{2}+\frac{a_{x\cdot 1y\cdot z}\,l_{1y}}{2l_y}-\frac{a_{xy\cdot 1z}\,l_{1z}}{2l_z}$. But this expression may be reduced to $\frac{v}{2}[1-r(a_x+2a_y-2a_{xy}-a_{yz}+a_{xyz})+a_{xy}-a_{xz}]-\frac{v(a_{x1\cdot y}-a_{x1\cdot z})\,l_{x1}}{2l_x}+\frac{1}{2l_y}[v(1+a_{y1\cdot z})l_{y1}-(a_{1y\cdot z}-a_{x\cdot 1y\cdot z})\,l_{1y}]-\frac{a_{xy\cdot 1z}\,l_{1z}}{2l_z}$; and which I shall denote by **M**. Consequently, when A is the oldest life, the required value will be represented by $s\mathbf{M}$.

§ 333. *Case* 2. Let B be the oldest of the three lives. In this case the sum of the first m terms of the several collateral series above alluded to will be equal to $\mathbf{M}-\frac{v(1-ra_{x|m})}{2}\cdot\frac{l_{x|m}\,v^m}{l_x}+\frac{v(1+a_{xz|m})}{2}\times\frac{l_x\,l_{z|m}\,v^m}{l_x\,l_z}-\frac{v(1+a_{x1\cdot z|m})}{2}\cdot\frac{l_{x1}\,l_{z|m}\,v^m}{l_x\,l_z}$. But, after the decease of B, the sum of the expectations for all the subsequent years will be more correctly expressed, as in the second case to Prob. XL., by $s(1-f)\,v(1-ra_{x|m})\,\frac{l_{x|m}\,v^m}{l_x}$; and which being added to the sum of the first m terms of the several collateral series above mentioned, will make the total present value of the given sum, when B is the oldest life, equal to $s(\mathbf{M}-k)$.

§ 334. *Case* 3. Let C be the oldest of the three lives. In this case the first m terms of the several collateral series above alluded to, will be equal to $\mathbf{M} - v(1+ra_{x|m})\frac{l_{x|m}v^m}{l_x} - v(1-ra_{y|m})\frac{l_{y|m}v^m}{l_y} + \frac{v(1+a_{xy|m})}{2}\frac{l_x l_{y|m} v^m}{l_x l_y} - \frac{a_{xy|m} l_x l_{y|m} v^m}{l_x l_y} + \frac{v(1+a_{x1\cdot y|m})}{2}\cdot\frac{l_{x1} l_{y|m} v^m}{l_x l_y}$. But, after the decease of C, the chance of receiving the given sum at the end of any subsequent year will depend on the happening of three different events: 1. that A and B both fail in the year; 2. that only B fails in the year, A having died in either of the preceding years; 3. that only A fails in the year, B having died after C in either of the preceding years. The probabilities of the happening of these several events in the $(m+1)$st year are respectively $\frac{d_x d_{y|m}}{l_x l_y}$, $\frac{d_{y|m}}{l_y}(1-\frac{l_{x|m}}{l_x})$ and $\frac{d_{x|m}}{l_x}(1-f-\frac{l_{y|m}}{l_y})$: which being added together and multiplied by sv^{m+1} will give the expectation of receiving the sum at the end of the $(m+1)$st year.

In like manner we may find the $(m+2)$nd, and all the subsequent expectations to the utmost extremity of human life: the sum of all which will be equal to s multiplied into $(1-f)\frac{v(1-ra_{x|m})}{2}\cdot\frac{l_{x|m}v^m}{l_x} + v(1-ra_{y|m}) \times \frac{l_{y|m}v^m}{l_y} - v(1+a_{xy|m})\frac{l_x l_{y|m} v^m}{l_x l_y} + \frac{a_{xy|m} l_x l_{y|m} v^m}{l_x l_y}$; and which being added to the first m terms of the several collateral series above found, will make the total present value of the given sum, when C is the oldest life, equal to $s(\mathbf{M}+u)$.

COROLLARY.

§ 335. When all the lives are equal, or of the same age A, the last three quantities in the formula denoted by **M** destroy each other, and the general expression in this case will become $\frac{sv}{2}[1 - r(3a_x - 3a_{xx} + a_{xxx})]$; that is, equal to half the present value of the given sum, to be received on the extinction of the longest of the three lives.

SCHOLIUM.

§ 336. Since the present values of the sum, as found by the preceding and the present problem, are together equal to $sTv[1-r(a_x+a_y-a_{xy}]$; or to the present value of the given sum to be received on the extinction of the longest of the two lives, A, B; it follows, that the present value of one of them being found, the present value of the other may be easily determined by subtracting such value from the general expression here given.

CHAPTER IX.

ON M. DE MOIVRE'S HYPOTHESIS.

§ 337. If the decrements of life, in any table of observations, were equal and uniform from birth to the extent of human life, the number of persons living at every succeeding age, as shown by that table, would be the terms of a decreasing arithmetical progression. Now, though this fact does not take place throughout the *whole* duration of life, yet, in several of the tables of observations (particularly in the *Northampton* table), it will be seen that, towards the middle ages, the decrements are, for many years together, constant and uniform, or at least very nearly so; and it was upon this view of the subject that M. De Moivre founded his ingenious hypothesis *that the decrements of life are in arithmetical* progression.[1] For he imagined that the deficiencies in one part of his scale would be compensated by the excesses of the other part: and that though the values deduced from this principle might not be strictly true, yet the error could not be very material. In order to accommodate this hypothesis to the method of determining the value of life annuities, it was necessary to suppose the extent of life confined to a certain period of time, which he (for reasons which subsequent observations have shown not to be sufficiently well founded) has taken at 86: remarking, however, at the same time, that the instances of persons living beyond that period are so few as not to be worth notice in such a general view of the subject.

Much obloquy has of late years been thrown on this hypothesis by Dr. Price and Mr. Morgan; the latter of whom has been particularly severe in his comments on its use and application. It is true that more recent discoveries have shown that it cannot always be depended upon: and the great share which these gentlemen have had in deducing the values of annuities from *real observations*, and thereby superseding the use of the hypothesis, may be some extenuation for the high tone they have assumed on this occasion. Nevertheless, the hypothesis itself is *still* of great and extensive use in the doctrine of annuities, and will ever remain a monument of the ingenuity and abilities of its illustrious inventor.

§ 338. The reader must be readily aware that the great advantage of M. De Moivre's hypothesis is the remarkable facility which it affords for determining the present value of annuities on any single or joint lives. The method of deducing such values from the first problem of the present work (see § 31) is extremely laborious, since the numerators of the fractions, which form the series there given, do not decrease in any regular manner. But when such numerators are in arithmetical progression, the sum of the whole series may be expressed in a general formula; and its value may then be found almost as soon as one term only of the other series just alluded to.

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[1] The same author also assumed as an hypothesis that the decrements of life were in a *geometrical* ratio: a principle which he applied to the valuation of annuities on *joint* lives. But this theory is too erroneous to be insisted on.

Although M. De Moivre, in the application of his principles to practical purposes, has assumed 86 as the extreme period of human life (that is, that out of eighty-six persons born, *one* of them will die annually till they are all extinct), yet this number does not necessarily follow from his hypothesis: for any other number, which might be found more agreeable to real observations, may be substituted in lieu thereof. Assuming, however, the same period as De Moivre, it is evident that the number of persons living at any age will be equal to the number of years between that age and 86: this number he calls the *complement* of life. Consequently, a, b, c, &c., will, upon this hypothesis, be equal to the complement of the lives A, B, C, &c., respectively; or equal to the difference between their respective ages and 86.

§ 339. This being premised, it will be readily seen that the expression $\frac{1}{l_x}(vl_{x1}+v^2\,l_{x2}+v^3\,l_{x3}+\&c.)$, which, by Prob. I. cor. 1, denotes the value of an annuity on the life A, will become $\frac{1}{a}[v(a-1)+v^2(a-2)+v^3(a-3)+\ldots v^a\,(a-a)]$. Now, this series may be divided into two others, the first of which is $[v+v^2+v^3+\ldots v^a]$; and the second, which is to be subtracted, is $\frac{1}{a}[v+2v^2+3v^3+\ldots av^a]$. But the former of these, which I shall denote by p, is equal to the present value of an annuity certain for the term a; and the latter is equal to $\frac{1}{a}\left[\frac{1+p-(a+1)\,v^a}{r}\right]$:[1] consequently the total present value of the annuity is, upon this hypothesis, equal to $p-\frac{1+p-(a+1)\,v^a}{ra}=\frac{a-(1+r)p}{ra}$.[2]

§ 340. In like manner, the expression $\frac{1}{l_x\,l_y}[vl_x\,l_{y|1}+v^2\,l_x\,l_{y|2}+v^3\,l_x\,l_{y|3}+\ldots]$, which denotes the value of an annuity on the two joint lives A, B, will become upon this hypothesis equal to $\frac{1}{ab}[v(a-1)\,(b-1)+v^2\,(a-2)$

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[1] See my *Doctrine of Interest and Annuities*, p. 89, where the method of summing this and the following series is pointed out.

[2] See Dodson, vol. ii. Ques. 56; Simpson, Prob. 1, cor. 5; De Moivre, p. 310.

$(b-2)+v^3(a-3)(b-3)+\ldots v^b(a-b)(b-b)]$; a series which is to be continued for b terms only, B being the oldest of the two lives. But if the numerators of the fractions composing this series be expanded by multiplication, the whole expression may be divided into three other distinct series: viz., $[v+v^2+v^3+\ldots v^b]-\frac{a+b}{ab}(v+2v^2+3v^3+\ldots bv^b]+\frac{1}{ab}\times$ $[v+4v^2+9v^3+\ldots v^b b^2]$. The first of these is equal to the present value of an annuity for the term of b, and I shall denote it by p; the second is (from what has been said above) equal to $-\frac{a+b}{ab}\times\frac{1+p-(b+1)v^b}{r}$; and the third is equal to $\frac{1}{abr}[1+p+2\times\frac{1+p-(b+1)v^b}{r}-(b+1)^2v^b]$: consequently the total present value of the annuity for two joint lives is equal to $\frac{1}{r}-\frac{1+r}{ar}\times\left[\frac{p}{b}(a-b-1-\frac{2}{r})+\frac{2}{r}\right]$.[1]

§ 341. By a similar method of reasoning it will be found that the expression $\frac{1}{l_x l_y l_z}[vl_x l_y l_{z|1}+v^2 l_x l_y l_{z|2}+v^3 l_x l_y l_{z|3}+\ldots]$, which denotes the value of an annuity on the three joint lives A, B, C, will become, upon this hypothesis, equal to $\frac{1}{abc}[v(a-1)(b-1)(c-1)+v^2(a-2)(b-2)(c-2)$ $+v^3(a-3)(b-3)(c-3)+\ldots v^c(a-c)(b-c)(c-c)]$; a series which is to be continued for c terms only, C being now the oldest life. But the whole of this series may, by expansion, be converted into the four following series: viz., $[v+v^2+v^3+\ldots v^c]-\frac{ab+ac+bc}{abc}(v+2v^2+3v^3+\ldots cv^c)$

$$+\frac{a+b+c}{abc}(v+4v^2+9v^3+\ldots c^2v^c)-\frac{1}{abc}(v+8v^2+27v^3+\ldots c^3v^c).$$

Now the first of these series is equal to the present value of an annuity certain for the term c, and I shall denote it by p, and the remaining ones may be summed in the same manner as in the preceding cases: whence the total present value of an annuity on the three joint lives is equal to

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[1] See Dodson, vol. ii. Ques. 64; Simpson, Prob. 1, cor. 5; Price, Note (L). I shall here take opportunity of observing that the strictures of Dr. Price on M. De Moivre's *rules* for calculating the value of annuities on joint lives (inserted in the fourth chapter of his *Observations on Rev. Pay.*,) allude altogether to an *approximation* given by De Moivre as deduced from his hypothesis of a geometrical ratio, and not to the *correct* expression. The formula in the text will oftentimes give the same values as those deduced from the *Northampton* Table, as may be seen by a comparison with the tables at the end of the present work; and I shall show in the sequel that these formulæ may be used with good effect in determining, with a tolerable degree of correctness, the value of annuities deduced from the *real probabilities* of life. See § 346.

$$\frac{1}{r}-\frac{1+r}{abr}\left[\frac{1}{r}(2b+2a-c-3-\frac{6}{r})-\frac{2p}{cr}(b+a-2c-3-\frac{3}{r})+\frac{p}{c}(b-c-1)(a-c-1)\right].$$[1]

§ 342. But since the publication of so many accurate tables of the values of annuities, deduced from *real observations*, these formulæ have become of little or no use, and are seldom resorted to, unless it is required to find a near value of an annuity on any given lives whose ages are not inserted in those tables. Nevertheless the hypothesis itself is still of great use in the doctrine of annuities, and facilitates very much several calculations arising from this subject, particularly in those cases where the contingency continues for a given *term* only, as will evidently appear from the following problem:—

PROBLEM LI.[2]

§ 343. To find the value of an annuity on a given life, during part of whose existence the decrements of life may be considered as equal.

SOLUTION.

Let A be the given life, and m the interval of equal decrements; also, let l_x be the number of persons living at the age of A, $l_{x|m}$ the number of persons living at an age m years older; and let the uniform decrements of life be denoted by δ. Then will the value of an annuity on this life be denoted by the series $\frac{1}{l_x}[v(l_x-\delta)+v^2(l_x-2\delta)+v^3(l_x-3\delta)+\ldots v^m(l_x-m\delta)+v^{m+1}\,l_{x1|m}+v^{m+2}\,l_{x2|m}+v^{m+3}\,l_{x3|m}+\ldots]$, which may be divided into two parts: viz., the series $\frac{1}{l_x}[v(l_x-\delta)+v^2(l_x-2\delta)+v^3(l_x-3\delta)+\ldots v^m(l_x-m\delta)]+\frac{1}{l_x}[v^{m+1}\,l_{x1|m}+v^{m+2}\,l_{x2|m}+v^{m+3}\,l_{x3|m}+\ldots]$. The latter of these is, by Prob. I. cor. 3, equal to $a_{x(m)}$, or equal to the value of an annuity on the life A *deferred* for m years; and the former may be divided into two others; viz., $[v+v^2+v^3+\ldots v^m]-\frac{\delta}{l_x}[v+v^2+v^3+\ldots v^m m]$: the sum of which may be readily found, from what has preceded, to be equal to p—

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[1] See Dodson, vol. ii. Ques. 69; Simpson, Prob. 1, cor. 5.

[2] De Moivre, p. 341; Simpson, Prob. 1, cor. 8; Price, Note (N); Dodson, vol. ii. Ques. 68. This last author has also given a method of solution (Ques. 61), provided the decrements of life be divided into *several* arithmetical progressions, each differing from one another.

$\frac{\delta}{l_x r}[(1+r)p-mv^m]$. Consequently the total value of the annuity will be $p-\frac{\delta}{l_x r}[(1+r)p-mv^m]+a_{x(m}$. But since δ is always equal to $\frac{l_x-l_{x|m}}{m}$;[1] and since $a_{x(m}$ is equal to $a_{x|m}\frac{l_{x|m}\,v^m}{l_x}$; this expression may be rendered more convenient for practice by making those substitutions: whence the value will become $\frac{1}{r}-\frac{(l_x-l_{x|m})(1+r)p}{l_x mr}-\frac{l_{x|m}\,v^m}{l_x r}+a_{x|m}\times\frac{l_{x|m}\,v^m}{l_x}=\frac{1}{r}-\frac{1}{l_x}\left[\frac{(l_x-l_{x|m})(1+r)p}{mr}-l_{x|m}\left(\frac{1}{r}-a_{x|m}\right)v^m\right]$. Now it appears by the *Northampton* table of observations that the decrements of life are *nearly* uniform during the whole interval from the age of 20 to the age of 80: and therefore the value of an annuity on a life of any intermediate age may be easily deduced from the formula here given, and will agree very nearly with the true values.

Example.—Let the rate of interest be 5 per cent., in which case the value of an annuity on a life of 80 years of age will be 3·515: and if from this we wish to deduce the value of an annuity on a life of 20 years of age, the formula will become $20-\frac{1}{5132}\left[\frac{5132-469}{60\times{\cdot}05}\times1{\cdot}05\times18{\cdot}9293-469\times(20-3{\cdot}515)\times{\cdot}0535\right]=14{\cdot}061$, and which is very near 14·007, or the true value as shown by the tables. Now when it is considered that the value here found could not have been deduced from the corollary in § 32, without the actual calculation of *sixty* terms of the series there given, the utility of this formula will be manifest.

COROLLARY.

§ 344. If the value of the annuity is required for a given term (m), and that term happens to be wholly within the interval of equal decrements, as shown by any given table of observations; the series $\frac{1}{l_x}\times[v(l_x-\delta)+v^2(l_x-2\delta)+v^3(l_x-3\delta)+\ldots v^m(l_x-m\delta)]=p-\frac{l_x-l_{x|m}}{l_x mr}[p(1+r)-mv^m]$ will denote the *exact* value in such case; and is a formula of considerable utility when we are not possessed of any tables of the value of annuities deduced from such observations. Or, if the decrements are very nearly equal, the formula will not differ materially from the true expression.

Example. Suppose it were required to find the value of an annuity for twenty years on a life aged 20, reckoning interest at 5 per cent. and the probabilities of living as at *Northampton*. In this case the formula would become $12{\cdot}4622 - \frac{5132 - 3635}{5132 \times 20 \times {\cdot}05} \times [1{\cdot}05 \times 12{\cdot}4622 - 20 \times {\cdot}3769] =$ 10·844 : which is the *correct* value of a temporary annuity for twenty years on a life aged 20; because it will be seen, by Table VII., that from the age of 20 to the age of 40 the decrements of life are equal. The value, deduced from the rule in page 46, is 10·847.

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[1] This may generally be assumed as the true value, without material error, even when the decrements are not exactly regular.

§ 345. This problem and its corollary will serve to show the useful purposes to which M. De Moivre's hypothesis may be occasionally appropriated, and the method of applying it whenever an opportunity occurs. But, since the decrements of life are most irregular in the younger and in the latter periods of existence, and are uniform (or nearly so) during the middle ages only, it will be found that this hypothesis cannot in all cases be safely used, unless in deducing the value of annuities or assurances for *terms*. In this respect it is of singular utility, and will be often found to save a laborious calculation, as I have already pointed out in the notes to some of the preceding problems; and therefore it will be unnecessary to enlarge more upon the subject in this place. It is in this manner that Mr. Morgan has condescended to use it: but it is done clandestinely, and (I know not for what reason) by a previous *denial* of the fact.[1]

I would here observe that the rule for determining the value of such annuities as depend on the whole continuance of any number of lives out of any other number of lives, or such as are in reversion or depending upon survivorships, and in general for determining such problems as are contained in the second, third, and fourth Chapters of this work, are the same on M. De Moivre's hypothesis as where they are deduced from real observations. For, in each case, solutions to such questions are obtained from tables which show the values of annuities on single and joint lives: and therefore the merit or demerit of M. De Moivre's hypothesis, as far as regards the value of *annuities*, will rest on the fundamental propositions above given.

But in deducing the value of *assurances*, or reversionary sums, his rules are certainly much more simple than when deduced from real observations: and it would be a fortunate circumstance if his hypothesis could be depended upon throughout the *whole duration* of life. As this, however, is not the case, we must be content with the facility which it oftentimes affords us of determining, in many cases, a very near value of such assurances for *terms*, as I have already fully explained in the sixth and eighth Chapters.[2]

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[1] See Price's *Obs. on Rev. Pay.*, vol. i. p. 61, Note (*e*); also § 236 of the present work.

[2] See §§ 184, 235; note 2, p. 127, and note 3, p. 132.

On the Method of Approximating to the Value of Life Annuities.

§ 346. I cannot dismiss this chapter, however, without noticing the utility and convenience of the formulæ arising from M. De Moivre's hypothesis, in enabling us also to deduce (from the values of annuities on single or joint lives at any one rate of interest) the values of annuities on the same lives, at any other rate of interest. In order to explain this method, I would observe that, according to M. De Moivre's hypothesis, the *expectation* of any life is equal to half the complement of such life :[1] consequently, the complement of any life is equal to twice its expectation. If, therefore, we substitute *twice* the *expectation* of any life deduced from real observations instead of the quantities a, b, or c, in the general formulæ in § 339 and § 340, the values thence arising will, in most cases, be much nearer the true values of annuities deduced from such observations, than when a, b, or c, is taken equal to the complement of such life according to M. De Moivre's hypothesis.[2] Or (which is all that is required in the present instance), the difference between the values of an annuity on any single or joint lives, deduced in this manner from the expectations of life, at any two rates of interest, will be nearly the same as the difference between the *correct* values of a similar annuity, at the same rates of interest, deduced from *real observations*. Consequently, when the value of an annuity according to any one rate of interest is given, we may readily obtain a very near value of a similar annuity at any other rate, by means of the first difference here alluded to, as will be evident from the following general rule.

Call the *correct* value, computed at any one rate of interest, the FIRST value.

Call the value, deduced from the *expectations* of life, at the *same* rate of interest, the SECOND value.

Call the value, deduced from the *expectations* of life, at any *other* rate of interest, the THIRD value.

Then the difference between the SECOND and THIRD values subtracted from, or added to, the FIRST value (according as the SECOND is greater or less than the THIRD) will be the near value of the annuity at the *other* rate of interest required.

Example 1. What is the near value of an annuity on a life aged 20 years at 4 per cent. interest, deduced from the correct value at 5 per cent., and according to the *Northampton* observations?

The *first*, or correct value at 5 per cent., is, by Table X., equal to 14·007. The second value, deduced at the same rate of interest from the expectation of that life by the formula in § 339, is equal to $\frac{66{\cdot}86-1{\cdot}05\times 19{\cdot}234}{66{\cdot}86\times{\cdot}05}=$ 13·959.[1] The *third* value, deduced from the same formula, at 4 per cent., is equal to $\frac{66{\cdot}86-1{\cdot}04\times 23{\cdot}183}{66{\cdot}86\times{\cdot}04.}=15{\cdot}984$. Therefore, $14{\cdot}007+(15{\cdot}984-13{\cdot}959)=16{\cdot}032$ will be the near value required; and which differs only a unit in the last figure from the true value, as given in Table X.

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[1] See the note in p. 43.

[2] See the observations at the end of the following note.

§ 347. The same principles will apply to the case of two *joint* lives; and it will be found that in both cases the deduced values are sometimes nearly the same as the correct values; that, generally, they do not differ more than a 20th or 30th part of a year's purchase; that in *joint* lives they differ less than in single lives; and that they come equally near to each other *whatever* the rates of interest are.[2]

On the value of increasing Life Annuities.

§ 348. The hypothesis of M. de Moivre furnishes us likewise with a convenient and useful formula for determining the value of *increasing* life annuities; that is, of £1, £2, £3, &c. (or any multiples of those sums), payable at the end of one, two, three, &c., years respectively, if the given life A be then in existence. For, the series expressing such value will (from what has been said in § 339) be evidently equal to $\frac{1}{a}\left[v(a-1)+v^2 2\,(a-2)+v^3 3\,(a-3)+\ldots v^a a\,(a-a)\right.$, and which may be divided into the two following series: $[v+2v^2+3v^3+\ldots av^a]-\frac{1}{a}\,1^2v+2^2v^2+3^2v^3+\ldots a^2v^a]$. In order to abridge the subsequent process, let us make

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[1] When twice the expectation is equal to a whole number with a decimal added (as is commonly the case), the value of an annuity for that term may be best computed in the following manner:—Suppose the number of years (as in the present case) to be 66·86, the value of an annuity for sixty-six years is, by Table IV., equal to 19·201; and the value of an annuity for sixty-seven years is 19·239. The difference between these two values is ·038; which, being multiplied by the decimal ·86, and the product ·033 added to the least of the two values, will give 19·234 for the value of the annuity for 66·86 years.

The second and third values here obtained (that is, 13·959 and 15·984) will be found, on a comparison with the values in Table X., to be much nearer the *true* values than those obtained from M. de Moivre's hypothesis. Consequently, this first step of the process will show that M. de Moivre's *formula* (as given in § 339) may sometimes be applied, with good effect, to find, in an expeditious manner by *one* operation, a near value of an annuity deduced from real observations.

[2] See a variety of examples, in proof of these assertions, in Dr. Price's *Obs. on Rev. Pay.*, vol. i. p. 231. The same author remarks, that "these deductions, in the case of single lives particularly, are so easy, and give the true value so nearly, that it will be scarcely ever *necessary* to calculate the exact values (according to any given observations) for more than *one* rate of interest." But, however convenient the above rules may be in our present dearth of useful tables, they by no means remove the necessity of calculating, at *several* rates of interest, any new tables that may be hereafter formed; and I should hope that no one, who may at any future time undertake this laborious task, will be influenced by so weak and so ill-judged an excuse. The object and real utility of tables of any kind is to save time and labour, and to prevent the occurrence of errors.

$$v+v^2+v^3+\ldots\ldots v^a=p$$
$$v+2v^2+3v^3+\ldots\ldots av^a=k$$
$$1^2v+2^2v^2+3^2v^3+\ldots\ldots a^2v^a=h,$$

then will the sum of the two series above given, or the value of the annuity required, be denoted by $k-\frac{h}{a}$.

But, from what has been said in § 339, it will be seen that the value of an annuity on a single life A is denoted by $p-\frac{k}{a}$; and from what has been said in § 340, that the value of an annuity on two equal joint lives AA is denoted by $p-\frac{2k}{a}+\frac{h}{aa}$. Therefore $a_x-a_{xx}=p-\frac{k}{a}-p+\frac{2k}{a}+\frac{h}{aa}=\frac{k}{a}-\frac{h}{aa}$; consequently $a(a_x-a_{xx})=k-\frac{h}{a}$ will be the value of the increasing annuity required: whence we deduce the following rule:—

§ 349. *From the value of an annuity on the given life subtract the value of an annuity on two equal joint lives of the same age with the given life; multiply the difference by the complement (or twice the expectation) of the given life: and the product will be the answer required.*

§ 350. *Example.* Let the given life be 40 years of age; and let the annuity be £1 the first year, £2 the second year, £3 the third year, and so on, according to the order of the natural numbers: what is the present value of this annuity, reckoning interest at 4 per cent., and the probabilities of living as observed at *Northampton?*

Here we shall have $a_x=13{\cdot}197$, $a_{xx}=9{\cdot}820$, and a (or twice the expectation) $=46{\cdot}16$; consequently $46{\cdot}16\times(13{\cdot}197-9{\cdot}820)=155{\cdot}882$ will be the value required.

If the annuities had been £10, £20, £30, &c., the present value would be $155{\cdot}882\times10$. Or, if they had been £15, £30, £45, &c., the present value would be $155{\cdot}882\times15$.

But, if the annuity *commences* with a larger sum than £1, and yet increases only by £1 in every year, we must add to the value above found, the value of an annuity on the given life multiplied into the first payment lessened by unity; and the sum will be the answer. Thus, if the annuities in the first case mentioned above, had been £15, £16, £17, &c., we must multiply 13·197 by 14; and the product, or 184·758, added to 155·882, will give 340·640 for the answer in this case required.

§ 351. These and many other instances (in addition to those already mentioned in various parts of this treatise) might be adduced to show the great utility and convenience of M. De Moivre's hypothesis in a general point of view. The most common cases will convince us that it may always lay claim to a considerable share of merit; but that it is particularly entitled to our approbation in enabling us to conduct our inquiries into many branches of this science, where the common analysis is not only exceedingly intricate, but sometimes entirely fails; and that it is by no means deserving of the false and ignominious epithets of "wretched" or "absurd."

CHAPTER X.

ON THE VALUE OF ANNUITIES PAYABLE HALF-YEARLY, ETC.; ON HALF-YEARLY, ETC. ASSURANCES; AND ON ANNUITIES SECURED BY LAND.

§ 352. In the preceding chapters, the values of annuities have been deduced on the supposition that they are all payable yearly: this is the most usual case. But, as others may occasionally occur, it will be useful to know the limits of the differences which arise in those cases: therefore, that nothing might be wanting on this subject, I shall make no apology for introducing the following investigations.[1]

If l_x, l_{x1}, l_{x2}, l_{x3}, etc., represent the number of persons living at the age of A, and at the age of 1, 2, 3, etc., years older than A, agreeably to what has been said in § 23, then will $\frac{d_x}{2}$, $\frac{d_{x1}}{2}$, $\frac{d_{x2}}{2}$, etc., denote the number of persons living at the end of ½, 1½, 2½, etc., years from the age of A: which, though perhaps not in *all* cases strictly true, will serve our present purpose, and be as near the real value as we could hope for. Consequently, the present value of an annuity on the life A, payable *half-yearly* is equal to $\frac{1}{2l_x}\left[\frac{v^{\frac{1}{2}}d_x}{2}+v\,l_{x1}+\frac{v^{\frac{3}{2}}d_{x1}}{2}+v^2\,l_{x2}+\frac{v^{\frac{5}{2}}d_{x2}}{2}+v^3\,l_{x3}+\text{etc.}\right]$:[2] a series which may evidently be divided into the two following ones: viz., $\frac{1}{2l_x}\left[\frac{v^{\frac{1}{2}}d_x}{2}+\frac{v^{\frac{3}{2}}d_{x1}}{2}+\frac{v^{\frac{5}{2}}d_{x2}}{2}+\text{etc.}\right]+\frac{1}{2l_x}[vl_{x1}+v^2l_{x2}+v^3l_{x3}+\text{etc.}]$. The latter of these is equal to $\frac{a_x}{2}$; and the former, which may be divided again into the two following series: $\frac{v^{\frac{1}{2}}}{4l_x}[l_x+vl_{x1}+v^2l_{x2}+v^3l_{x3}+\text{etc.}]+\frac{(1+r)^{\frac{1}{2}}}{4l_x}[vl_{x1}+v^2l_{x2}+v^3l_{x3}+\text{etc}]$, is equal to $\frac{v^{\frac{1}{2}}(1+a_x)}{4}+\frac{(1+r)^{\frac{1}{2}}a_x}{4}$. Whence the total present value of the annuity is equal to $\frac{a_x}{2}+\frac{v^{\frac{1}{2}}(1+a_x)}{4}+\frac{(1+r)^{\frac{1}{2}}a_x}{4}=\frac{v^{\frac{1}{2}}}{4}\times[2(1+r)^{\frac{1}{2}}a_x+1+a_x+(1+r)a_x]=\frac{v^{\frac{1}{2}}}{4}[2(1+r)^{\frac{1}{2}}+2+r]\times a_x+\frac{v^{\frac{1}{2}}}{4}$. But, since the quantity $\frac{v^{\frac{1}{2}}}{4}[2(1+r)^{\frac{1}{2}}+2+r]$ seldom much exceeds unity,[1] this expression may be taken (without material error) equal to $a_x+\frac{v^{\frac{1}{2}}}{4}$; and, since $\frac{v^{\frac{1}{2}}}{4}$ is seldom much below $\frac{1}{4}$,[2] the expression may be still further reduced to $a_x+\frac{1}{4}$. That is, if to the value of the annuity payable yearly we add a *quarter of a year's purchase*, the sum will be *very near* the value of the same annuity payable *half-yearly*. The *exact* values, however, may be easily determined from the general expression above given.

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[1] A person who receives a life annuity *half-yearly* has a double advantage over the one who receives the same annuity yearly; for, besides the interest of each half-yearly payment for six months, he has a chance of receiving one half-year's payment *more* than if he were paid yearly. In like manner, a person who receives a life annuity *quarterly* has a double advantage over one who receives the same annuity half-yearly, &c. &c. See this subject detailed at full length in Baron Maseres's *Doctrine of Life Annuities*, pp. 233—260.

[2] The reader is supposed to be acquainted with the method of deducing the present value of an annuity *certain* payable half-yearly, &c., as explained in my *Doctrine of Interest and Annuities*, Chap. X.

§ 353. If we wish to determine the present value of a similar annuity payable *quarterly*, we must take $\frac{3l_x+l_{x1}}{4}$, $\frac{l_x+3l_{x1}}{4}$, $\frac{3l_{x1}+l_{x2}}{4}$, $\frac{l_{x1}+3l_{x2}}{4}$, $\frac{3l_{x2}+l_{x3}}{4}$, $\frac{l_{x2}+3l_{x3}}{4}$, &c., to denote the number of persons living at the end of $\frac{1}{4}$, $\frac{3}{4}$, $1\frac{1}{4}$, $1\frac{3}{4}$, $2\frac{1}{4}$, $2\frac{3}{4}$, &c., years:[3] whence the present value of an annuity on the life A payable *quarterly* will be equal to $\frac{1}{4l_x}\left[\frac{v^{\frac{1}{4}}(3l_x+l_{x1})}{4}+\frac{v^{\frac{1}{2}}(l_x+l_{x1})}{2}+\frac{v^{\frac{3}{4}}(l_x+3l_{x1})}{4}+vl_{x1}+\frac{v^{\frac{5}{4}}(3l_{x1}+l_{x2})}{4}+\frac{v^{\frac{3}{2}}(l_{x1}+l_{x2})}{2}+\frac{v^{\frac{7}{4}}(l_{x1}+3l_{x2})}{4}\right.$

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[1] When the rate of interest is 2 per cent. per annum, the quantity here alluded to is equal to 1·000025; and when the rate of interest is 10 per cent. per annum, it is equal to 1·000569; whence a judgment may be formed of its value at any intermediate rate.

[2] When the rate of interest is 2 per cent. per annum, this quantity is equal to ·2475; and when the rate of interest is 10 per cent. per annum, it is equal to ·2384; but it will be seen that as this quantity *decreases*, the one mentioned in the last note *increases*; whence $a_x+\frac{1}{4}$ will seldom much exceed the true value.

[3] These are the arithmetical means between the number of persons living at the age of A, and at the several ages of $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, &c., years older; and are sufficiently near for the purposes here intended.

$+v^2l_{x2}+\&c.\Big]$. But, this series may be divided into the four following ones: viz., $\frac{v^{\frac{1}{4}}}{16l_x}\Big[\frac{3l_x+l_{x1}}{1}+v(3l_{x1}+l_{x2})+v^2(3l_{x2}+l_{x3})+\ldots\Big]=$ $\frac{v^{\frac{1}{4}}}{16}3(1+a_x)+\frac{a_x(1+r)^{\frac{3}{4}}}{16}$; $\frac{v^{\frac{1}{2}}}{8l_x}\Big[\frac{l_x+l_{x1}}{1}+v(l_{x1}+l_{x2})+v^2(l_{x2}+l_{x3})+\ldots\Big]=$ $\frac{v^{\frac{1}{2}}(1+a_x)}{8}+\frac{a_x(1+r)^{\frac{1}{2}}}{8}$; $\frac{v^{\frac{3}{4}}}{16l_x}\Big[\frac{l_x+3l_{x1}}{1}+v(l_{x1}+3l_{x2})+v^2(l_{x2}+l_{x3})+\ldots\Big]=$ $\frac{v^{\frac{3}{4}}(1+a_x)}{16}+\frac{3a_x(1+r)^{\frac{1}{4}}}{16}$; and $\frac{1}{4l_x}(vl_{x1}+v^2l_{x2}+v^3l_{x3}+\ldots)=\frac{a_x}{4}$. Whence the total present value of the annuity is equal to $\frac{v^{\frac{1}{4}}3(1+a_x)}{16}+\frac{a_x(1+r)^{\frac{3}{4}}}{16}+\frac{v^{\frac{1}{2}}(1+a_x)}{8}+\frac{a_x(1+r)^{\frac{1}{2}}}{8}+\frac{v^{\frac{3}{4}}(1+a_x)}{16}+\frac{3a_x(1+r)^{\frac{1}{4}}}{16}+\frac{a_x}{4}=\frac{v^{\frac{3}{4}}}{16}\,[4(1+r)^{\frac{3}{4}}+(4+r)(1+r)^{\frac{1}{2}}+2(2+r)(1+r)^{\frac{1}{4}}+4+3r]\times a_x+\frac{v^{\frac{3}{4}}}{16}\,[3(1+r)^{\frac{1}{2}}+2(1+r)^{\frac{1}{4}}+1]$.

But since the fractional quantity by which a_x is multiplied seldom much exceeds unity,[1] this expression may be taken (without material error) equal to $a_x+\frac{v^{\frac{3}{4}}}{16}[3(1+r)^{\frac{1}{2}}+2(1+r)^{\frac{1}{4}}+1]$: and since $\frac{v^{\frac{3}{4}}}{16}[3(1+r)^{\frac{1}{2}}+2(1+r)^{\frac{1}{4}}+1]$ is seldom much below $\frac{3}{8}$,[2] the expression may be still further reduced to $a_x+\frac{3}{8}$. That is, if to the value of the annuity payable yearly we add *three-eighths of a year's purchase*, the sum will be *very near* the value of the same annuity payable *quarterly*. The exact values, however, may be easily determined, as in the former case, from the general expression above given.

§ 354. Upon M. De Moivre's hypothesis, the present value of a life annuity payable *half-yearly* will be denoted by the series $\frac{1}{2l_x}[v^{\frac{1}{2}}(a-\frac{1}{2})+v(a-1)+v^{\frac{3}{2}}(a-\frac{3}{2})+v^2(a-2)+v^{\frac{5}{2}}(a-\frac{5}{2})+v^3(a-3)+\ldots v^a(a-a)]$; which may be divided into the two following ones: viz., $\frac{1}{2}[v^{\frac{1}{2}}+v+v^{\frac{3}{2}}+v^2+\ldots av^a]-\frac{1}{4a}\,[v^{\frac{1}{2}}+2v+3v^{\frac{3}{2}}+4v^2+\ldots 2av^a]$. The first of these is equal to

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[1] When the rate of interest is 2 per cent. per annum, the quantity here alluded to is equal to 1·000037; and when the rate of interest is 10 per cent. per annum, it is equal to 1·00071; whence a tolerably correct opinion may be formed of its value at any intermediate rate.

[2] When the rate of interest is 2 per cent. per annum, this quantity is equal to ·3719; and when the rate of interest is 10 per cent. per annum, it is equal to ·3605; but as this value decreases, the one mentioned in the last note increases; whence $a_x+\frac{3}{8}$ will seldom much exceed the true value.

$\frac{1}{2} \times \frac{1-v^a}{(1+r)^{\frac{1}{2}}-1} = h$;[1] and the second is equal to $-\frac{1}{4a} \times \frac{2h(1+r)^{\frac{1}{2}}-2av^a}{(1+r)^{\frac{1}{2}}-1}$:[2] consequently, the total present value of the annuity is equal to $\frac{1}{2a} \times \frac{a-h(1+r)^{\frac{1}{2}}}{(1+r)^{\frac{1}{2}}-1}$.

In like manner, the present value of a life annuity payable *quarterly* will, upon the same hypothesis, be equal to $\frac{1}{4a}[v^{\frac{1}{4}}(a-\frac{1}{4})+v^{\frac{1}{2}}(a-\frac{1}{2})+v^{\frac{3}{4}}(a-\frac{3}{4})+v(a-1)+\ldots v^a(a-a)$; which may be divided into the two following series: $\frac{1}{4}[v^{\frac{1}{4}}+v^{\frac{1}{2}}+v^{\frac{3}{4}}+v+\ldots v^a]-\frac{1}{16a}[v^{\frac{1}{4}}+2v^{\frac{1}{2}}+3v^{\frac{3}{4}}+4v+\ldots 4av^a]$. But, the first of these series is equal to $\frac{1}{4} \times \frac{1-v^a}{(1+r)^{\frac{1}{4}}-1} = q$;[3] and the second is equal to $-\frac{1}{16a} \times \frac{4q(1+r)^{\frac{1}{4}}-4av^a}{(1+r)^{\frac{1}{4}}-1}$: consequently, the total present value of the annuity is in this case equal to $\frac{1}{4a} \times \frac{a-q(1+r)^{\frac{1}{4}}}{(1+r)^{\frac{1}{4}}-1}$.

By a similar process we might find the present value of such annuities payable at any other intervals; but it will be sufficient to show the extreme limit of the increase which arises from this, supposing the annuity to be payable at such smaller intervals; this limit takes place when the annuity is considered as being paid *momently*, in which case the expression becomes $\frac{v(a-m)}{a\text{NL}}$; where m is equal to $\frac{v(a-v^a)}{\text{NL}}$, or, to the present value of an annuity *certain* for the term a, payable *momently*.[4]

§ 355. If the numerical value of these expressions for half-yearly and quarterly annuities, according to the hypothesis of M. De Moivre, be compared with those deduced from real observations, they will be found to confirm the accuracy of each other, and to justify the rule which I have before given, namely, that the value of annuities payable *yearly* must be increased nearly $\frac{1}{4}$ of a year's purchase, in order to show the value of the same annuities payable *half-yearly;* and that they must be increased nearly $\frac{3}{8}$ of a year's purchase, in order to show the value of the same annuities payable *quarterly;* and also that they must be increased by $\frac{1}{2}$ a year's purchase in order to show the value of the same annuities payable *momently*.

§ 356. The reader must observe that, in all these cases, I have had regard only to the true rate of *annual* interest, agreeably to the principles which I have laid down in another work[1] for determining the value of annuities in general. But such annual rate must always be expressed in terms of the nominal rate, by making the substitutions there alluded to,[2] according as the interest is payable half-yearly, quarterly, &c.: whereby we shall find that, on M. De Moivre's hypothesis, the present value of an annuity, on the life whose complement is a, payable *yearly*, *half-yearly*, *quarterly*, and *momently*, and on the supposition that the *interest* is also payable at the *same* periods, will be denoted respectively by $\frac{a-(1+r)y}{ar}$, $\frac{a-(1+\frac{r}{2})h}{ar}$, $\frac{a-(1+\frac{r}{4})q}{ar}$, and $\frac{a-m}{ar}$: where y, h, q, and m will now respectively denote the present values of annuities certain for the term a, payable *yearly*, *half-yearly*, *quarterly*, and *momently*, and on the supposition that the *interest* also is payable at the *same* periods.

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[1] See my *Doctrine of Interest and Annuities*, p. 56.

[2] *Ibid*. p. 89.

[3] *Ibid*. p. 56.

[4] *Ibid*. p. 58. And I would here observe, that I take NL to denote the *Neperean* logarithm of the quantity immediately following it.

If the numerical value of these expressions be compared with each other, it will be found that the half-yearly annuities will, upon this principle, be about $\frac{2}{10}$ of a year's purchase, and quarterly annuities about $\frac{3}{10}$ of a year's purchase, more than the value of the same annuities payable yearly; and this is the rule given by Dr. Price for such purpose.[3] But as the periods of the payments of the *annuity* are totally independent of the periods of the payment of interest, and ought not to be confounded together (as I have more fully explained in the tenth chapter of my *Doctrine of Interest and Annuities*), we shall find that the addition to the tabular values of $\frac{1}{4}$ and $\frac{3}{8}$ respectively, as stated in pages 197 and 198, will be the most correct rule for general use; agreeably to what has been already advanced by Mr. Simpson in his *Doctrine of Annuities and Reversions*, page 79.

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[1] See my *Doctrine of Interest and Annuities*, p. 55.

[2] That is, by substituting $(1+\frac{r}{m})^m - 1$ for r; agreeably to what I have said in my *Doctrine of Interest and Annuities*, § 77, p. 54.

[3] See his *Observations on Rev. Pay.*, vol. i. p. 246; and the investigations which are there annexed. The two following examples, given by him, will show the real difference which arises in these cases:—

VALUE OF AN ANNUITY ON A SINGLE LIFE.
Interest 4 per cent.

Age.	Payable Yearly.	Payable Half-Yearly.	Payable Quarterly.	Payable Momently.
36	13·829	14·010	14·101	14·191
61	8·753	8·973	9·072	9·199

As the two methods of investigation are, however, now before the public, the computist may adopt that rule which he conceives to be best suited to the circumstances of the case. Dr. Price had endeavoured to overturn Mr. Simpson's rule without stating the grounds of his dissent, and to substitute his own without explaining the nature and cause of their difference.

§ 357. Hitherto I have considered the differences in the value of annuities on *single* lives only, but it will be evident that, on the supposition that money is improved at a given *annual* rate of interest, the differences will be nearly the same on two *joint* lives, whose value is deduced from real observations, and whose probabilities of living to the end of every half-year are respectively denoted by $\frac{l_x l_y + l_{x1} l_{y1}}{2 l_x l_y}$, $\frac{l_{x1} l_{y1} + l_{x2} l_{y2}}{2 l_x l_y}$, $\frac{l_{x2} l_{y2} + l_{x3} l_{y3}}{2 l_x l_y}$, &c.; which probabilities, though not strictly correct, may answer the present purpose.

On M. De Moivre's hypothesis, however, and on the supposition that money can be improved at interest payable at the *same* periods as the annuity, the value of an annuity on the two joint lives whose complements are a and b, payable *half-yearly*, is accurately expressed by the following series:

$$\frac{1}{2ab}\left[\frac{(a-\frac{1}{2})(b-\frac{1}{2})}{1+\frac{r}{2}}+\frac{(a-1)(b-1)}{(1+\frac{r}{2})^2}+\frac{(a-\frac{3}{2})(b-\frac{3}{2})}{(1+\frac{r}{2})^3}+\frac{(a-2)(b-2)}{(1+\frac{r}{2})^4}+\dots\frac{(a-b)(b-b)}{(1+\frac{r}{2})^{2b}}\right]$$

: the sum of which may, from what has preceded, be easily found equal to $\frac{1}{r}-\frac{2+r}{2ar}\left[\frac{h}{b}\times(a-b-\frac{1}{2}-\frac{2}{r})+\frac{2}{r}\right]$.

In like manner the value of an annuity on those lives, payable *quarterly*, is on the same principle equal to

$$\frac{1}{4ab}\left[\frac{(a-\frac{1}{4})(b-\frac{1}{4})}{1+\frac{r}{4}}+\frac{(a-\frac{1}{2})(b-\frac{1}{2})}{(1+\frac{r}{4})^2}+\frac{(a-\frac{3}{4})(b-\frac{3}{4})}{(1+\frac{r}{4})^3}+\frac{(a-1)(b-1)}{(1+\frac{r}{2})^4}+\dots\frac{(a-b)(b-b)}{(1+\frac{r}{4})^{4b}}\right]$$

: the sum of which is equal to $\frac{1}{r}-\frac{4+r}{4ar}\left[\frac{q}{b}(a-b-\frac{1}{4}-\frac{2}{r})+\frac{2}{r}\right]$.

And, if the annuity and the interest are both supposed to be payable *momently*, its value will come out equal to $\frac{1}{r}-\frac{1}{ar}\left[\frac{m}{b}(a-b-\frac{2}{r})+\frac{2}{r}\right]$. In which several formulæ, the quantities h, q, and m denote the present value of annuities certain for the term b, payable *half-yearly*, *quarterly*, and *momently*, on the supposition that the *interest* also is payable at the *same* periods respectively.

The following examples, given by Mr. Morgan, will show the real difference that arises in these cases:—

VALUE OF AN ANNUITY ON TWO JOINT LIVES.				
Ages.	Payable Yearly.	Payable Half-Yearly.	Payable Quarterly.	Payable Momently.
20–36	11·227	11·427	11·565	11·629
36–36	10·394	10·600	10·703	10·808
36–61	7·448	7·673	7·793	7·901
61–61	6·144	6·374	6·517	6·602

On Half-yearly, &c. Assurances.

§ 358. By a similar method of proceeding to that which has been adopted in the former part of this chapter, we might determine the present value of assurances for every *half*-year, *quarter*-year, &c., of human existence. For, if the number of persons living at the end of $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, &c., years from the age of A be denoted by the same quantities as in page 197, then will the probabilities of such life becoming extinct *in* the first, second, third, &c., *half*-years, be respectively represented by $\frac{d_x}{2l_x}$, $\frac{d_x}{2l_x}$, $\frac{d_{x1}}{2l_x}$, $\frac{d_{x1}}{2l_x}$, $\frac{d_{x2}}{2l_x}$, &c.; consequently the present value of an assurance of the sum s on the life A for every *half*-year of human existence will be truly expressed by $\frac{s}{2l_x}[v^{\frac{1}{2}}d_x+vd_x+v^{\frac{3}{2}}d_{x1}+v^2d_{x1}+v^{\frac{5}{2}}d_{x2}+\text{\&c.}]=\frac{s}{2}[1+(1+r)^{\frac{1}{2}}]\times v(1-ra_x)$.

§ 359. In like manner, if the number of persons living at the end of $\frac{1}{4}$, $\frac{3}{4}$, $1\frac{1}{4}$, &c., years from the age of A be denoted by the same quantities as in § 353, then will the probabilities of such life becoming extinct *in* the first, second, third, &c., *quarter*-years, be respectively represented by $\frac{d_x}{4l_x}$, $\frac{d_x}{4l_x}$, $\frac{d_x}{4l_x}$, $\frac{d_x}{4l_x}$, $\frac{d_{x1}}{4l_x}$, $\frac{d_{x1}}{4l_x}$, &c.; consequently the present value of an assurance of the sum s on the life A for every *quarter*-year of human existence will be truly expressed by $\frac{s}{4l_x}[v^{\frac{1}{4}}d_x+v^{\frac{1}{2}}d_x+v^{\frac{3}{4}}d_x+vd_x+v^{\frac{5}{4}}d_{x1}+v^{\frac{3}{2}}d_{x1}+\text{\&c.}]=\frac{s}{4}[1+(1+r)^{\frac{1}{4}}+(1+r)^{\frac{1}{2}}+(1+r)^{\frac{3}{4}}\times v(1-ra_x)=\frac{s}{4}\times\frac{r}{(1+r)^{\frac{1}{4}}-1}\times v(1-ra_x)$.

§ 360. By continuing these subdivisions, we shall find that the present value of the same assurance on the life A, for every nth part of a year, will be truly expressed by $\frac{s}{n}[1+(1+r)^{\frac{1}{n}}+(1+r)^{\frac{2}{n}}+\dots(1+r)^{\frac{n-1}{n}}]\times v(1-ra_x)=\frac{s}{n}\times\frac{r}{(1+r)^{\frac{1}{n}}-1}\times v(1-ra_x)$. Now, when n is infinite, this formula becomes equal to $\frac{srv}{\text{NL}}\times v(1-ra_x)$,[1] and which consequently denotes the value of the assurance for every *moment* of human existence; that is, the value of the given sum to be received *immediately* on the extinction of the given life.

Note.—Since the Neperean logarithm of $(1+r)$ differs but little from $\frac{2r}{2+r}$, this formula may be rendered more convenient for practice by means of the expression $\frac{sr(2+r)}{2r}\times v(1-ra_x)=s(1+\frac{r}{2})\times v(1-ra_x)$: which exceeds the value of a yearly assurance, deduced from the rule in page 95, by the quantity $\frac{sr}{2}\times v(1-ra_x)$.

It may be necessary to remark that these values are all deduced from the true *annual* rate of interest, which may be reduced to the *nominal* rate by making the substitution alluded to in the note to § 356.

On Life Annuities secured by Land.

§ 361. A life annuity, *secured by land*,[2] differs from that kind of life annuity which has been treated of in the preceding part of this work, inasmuch that if the annuitant dies at any time between the stated periods for the payment of the annuity, his heirs are to receive such a sum as will be proportional to the time elapsed between the last payment and his death; whereas, in all the cases hitherto considered, if the annuitant dies on the day preceding the time of payment, or sooner, his heirs cannot claim any part or portion of the annuity.

In this case, supposing the annuity payable yearly, the annuitant (since there is the same chance for his dying in one half of any year as in the other) may be considered as having an expectation of half a year's payment more than he would be otherwise entitled to. But the value of the half of £1, to be received on the extinction of any life A, is, by Prob. XXII., equal to $\frac{v}{2}(1-ra_x)$; and, which is the addition that ought to be made to the value of an annuity payable yearly, in order to obtain its value when secured by land: consequently the value of such annuity is $a_x+\frac{v(1-ra_x)}{2}=\frac{v}{2}[1+(2+r)a_x]$.

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[1] For, in such case $n[(1+r)^{\frac{1}{n}}-1]$ is equal to the Neperean logarithm of $(1+r)$. See Euler's Introd. in *Anal. Inf.*, vol. i. chap. 7, § 119, and also what has been said on this subject in my *Doctrine of Interest and Annuities*, p. 46.

[2] See on this subject, Dodson, vol. iii. Ques. 1 to 4, and 8 to 14; Price, vol. i. p. 244.

§ 362. In like manner, supposing the annuity payable half-yearly, the annuitant may be considered as having an expectation of a quarter of a year's payment more than he would be otherwise entitled to. But the value of the quarter of £1 to be received on the extinction of the life A in any half year, is, by the formula in page 202, equal to $\frac{v}{8}[1+(1+r)^{\frac{1}{2}}\times(r-a_x)]$; and, which is the addition that ought to be made to the value of an annuity payable half-yearly, in order to obtain its value when secured by land. And so on for the additions that ought to be made to the value of an annuity payable quarterly, &c. But the *difference* between the value of an annuity payable yearly, *not* secured by land, and the value of an annuity payable at the same, or at any other intervals, which *is* secured by land, can in no case exceed 0·5, or half unity.

§ 363. M. De Moivre, in his *Doctrine of Chances*, page 338, has given a theorem for finding the value of an annuity *secured by land* and payable yearly, which he deduced by a differential process—a method easily applicable to his hypothesis; and Mr. Dodson, in the third volume of his *Mathematical Repository*, page 4, has given another theorem for that purpose (obtained without the aid of that calculus), which brings out nearly the same answers.[1] But Mr. Simpson, in his *Select Exercises*, page 323, and in the *Supplement* to his Doctrine of Annuities, page 70, has given a theorem which shows the value, not of an annuity payable yearly and secured by land, but of an annuity payable *momently* at a given *annual* rate of interest.[2] The values in all these cases being obtained from M. De Moivre's hypothesis.

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[1] M. De Moivre's formula is $\frac{1}{\rho}-\frac{y}{a\times\text{NL.}(1+\rho)}$; where y denotes the value of an annuity *certain* for the term a, payable yearly. Now the Neperean logarithm of $(1+\rho)$ is very nearly equal to $\frac{2\rho}{2+\rho}$: if, therefore, we substitute this latter quantity instead of NL.$(1+\rho)$, the above formula will become $\frac{a-y(1+\frac{\rho}{2})}{a\rho}$; which is the same as that given by Mr. Dodson, and which exceeds the value of an annuity not secured by land (as deduced in page 189) by the quantity $\frac{y}{2a}$.

[2] Mr. Simpson's formula is $\frac{1}{\text{NL.}(1+\rho)}-\frac{m}{a\times\text{NL.}(1+\rho)}=\frac{a-m}{a\times\text{NL.}(1+\rho)}$; which is the same as that given in § 354, for determining the value of a life annuity payable *momently*, at a given *annual* rate of interest; but this is certainly not a correct mode of proceeding in order to find the value of an annuity *secured by land*.

Dr. Price is wrong in asserting that "Mr. Simpson makes the excess of the value of

I would here observe, that the formulæ, which I have given above, are the first that have been deduced from real observations, and are much more simple than those deduced from M. De Moivre's hypothesis. But though they readily follow, after the investigations that have been previously entered into, and might easily have been adapted by preceding writers to the value of annuities as deduced from such observations, yet those who have been the most forward to attack the whole of M. De Moivre's principles have not only suffered his formulæ on this, *and on other subjects*, to remain uncorrected and unreproved; but have inserted them in their works as affording a *proper* and *correct* solution to such cases !!!

CHAPTER XI.

ON THE VALUE OF DEFERRED ANNUITIES, REVERSIONARY ANNUITIES, AND ASSURANCES, IN ANNUAL PAYMENTS.

§ 364. In all those cases of deferred annuities mentioned in Prob. I. cor. 3, and in the corollaries to the subsequent problems, as well as in all cases of Assurances, I have deduced the values of the same in one *single* payment; but it is oftentimes required to determine such values in *annual* payments. The method of doing which I shall now proceed to show.

In the case of *Deferred* annuities, depending on any number of joint lives ABC, the value in one single payment is (by Prob. I. cor 3) denoted by $a_{xyz(m}$. Now, if the purchaser of this annuity is desirous of paying for the same by equal *annual* payments during the given term,[1] those equal payments ought to be such that their total present value shall be equal to the single payment above mentioned; or, in other words, he should pay instead of such sum an *equivalent annuity* during the given term.

§ 365. Let the required annual payment be denoted by p; and let the value of a temporary annuity on the given lives (that is, of an annuity to continue till the period when the deferred annuity commences) be denoted by $a_{xyz)m}$: then, since the value of the deferred annuity, or $a_{xyz(m}$, is to be paid for by equal annual payments during the time such annuity is deferred

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such an annuity above the value of an annuity payable yearly but not secured by land, *double* to the same excess derived from Mr. Dodson's and M. De Moivre's rules." The truth is, that not only Dr. Price, but Mr. Simpson himself, appear to have been deceived by the similarity of the symbols employed in the two formulæ compared, without sufficiently considering that those symbols denote *different* quantities.

[1] Such annual payments, however, subject to failure if the given lives become extinct before the end of that period.

(subject to failure if any of the given lives become extinct in that period), it is evident that the sum or value of such payments must be equal to the value of an annuity, on the given lives *for such time*, of the yearly value of p: that is, $pa_{xyz)m} = a_{xyz(m}$.

This, however, is on the supposition that the first annual payment is not made till the *end* of the first year, and continued at the *end* of every subsequent year till the expiration of the term. But this rarely, if ever, happens; and the usual, if not the invariable method, is to advance the first payment *immediately*, and the remaining ones at the *beginning* of each of the following years: so that the number of payments shall be equal to the number of years during which the annuity is deferred. Therefore (since the payment which was supposed, in the preceding case, to be made at the end of the term is now made at the *beginning*) we must add unity to the value of a temporary annuity for *one year less than*[1] the given term: and this quantity multiplied by the annual payment will be equal to the value of the deferred annuity. Consequently the formula will, in this case, become $p(1 + a_{xyz)m-1} = a_{xyz(m}$; whence $p = \frac{a_{xyz(m}}{1 + a_{xyz)m-1}}$,[2] and whence the following rule:—

§ 366. *Divide the value of the Deferred annuity, by unity added to the value of a similar Temporary annuity for one year less than the given term: the quotient will be the annual payment required.*

For examples of the use and application of this rule, see the Scholium to Question 6 in Chapter XII.

§ 367. The same rule will apply to the case of deferred annuities depending on the *longest* of two or more lives; see Prob. II. cor. 2. For, if the value of an annuity on the longest of any number of lives be denoted by L, then will the value of a similar *deferred* annuity be denoted by $L_{(m}$; and also the value of a similar *temporary* annuity for *one year less* than the given term will be denoted by $L_{)m-1}$. Consequently, from what has been above said, we shall have $p = \frac{L_{(m}}{1 + L_{)m-1}}$.

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[1] Dr. Price, in all the cases of annual payments which he has given, says that we must add unity to the value of a temporary annuity *for the given term*, by which means he makes the number of payments to be one more than ever occurs. The reader should particularly observe this in comparing *his* rules with the formula here given.

[2] Since $a_{xyz)m-1} = a_{xyz)m} - \frac{v^m l_x l_y l_{z|m}}{l_x l_y l_z} = a_{xyz} - a_{xyz(m} - \frac{v^m l_x l_y l_{z|m}}{l_x l_y l_z}$, it follows that the formala given in the text may be denoted by $\frac{a_{xyz(m}}{1 + a_{xyz} - (a_{xyz(m} + \frac{v^m l_x l_y l_{z|m}}{l_x l_y l_z})}$, which will be oftentimes found very convenient in practice.

For examples of the use and application of this formula, see Question 11 in Chapter XII.

In this case, however, it should be particularly observed that if the deferred annuity depends on the *joint* continuance of the given lives to the end of the given term (as mentioned in Prob. II. cor. 3) the formula will become $p = -\frac{L_{(m}}{1+a_{xyz)m-1}}$.

For examples of the use and application of this formula, see the Scholium to Question 11 in Chapter XII.

§ 368. A similar method of reasoning will lead us to the true value, in annual payments during the continuance of the given lives, of any *Reversionary* annuity. Thus, let the value of the reversionary annuity, mentioned in the first case in page 54, be denoted by a_r; then will the value of the same in annual payments, during the joint continuance of the two lives, be $p = \frac{a_r}{1+a_{xp}}$. The same formula extends also to the case of *Deferred* reversionary annuities.

For examples of the use and application of this formula, see Scholium to Question 13 in Chapter XII.; and also Question 18, and the Scholium to Question 18 in that chapter.

But, if the reversionary annuity be *Temporary*, or for a given term only, and such annuity be denoted by $a_{r)m}$, we shall have $p = \frac{a_{r)m}}{1+a_{xp)m-1}}$.

For examples of the use and application of this formula, see Question 19 in Chapter XII.

§ 369. The principles here laid down will likewise extend to all the cases of *Assurances* mentioned in Chapter VI.; whether for the whole continuance of life, or for any given term. For, if the present value of an assurance of any given sum be denoted by A_s, and the present value of a *temporary* assurance of a similar sum be denoted by $A_{s)m}$, then will the equivalent annuity during the joint continuance of all the lives involved be, in the first case, $p = \frac{A_s}{1+a_{xyz}}$; and in the latter case, $p = \frac{A_{s)m}}{1+a_{xyz)m-1}}$.

It is scarcely necessary to observe, when A_s denotes the value of an assurance on the *longest* of any number of lives, that a_{xyz} will in such case denote the value of an annuity on the *longest* of such lives; agreeably to what has been said in Prob. XXII. cor. 2. And so likewise of any other assurance there alluded to.

For the use and application of the formula, see Questions 26, 27, and 29 in Chapter XII.

§ 370. With respect to those assurances which are the subject of Chapter VIII., the annual payment may be divided into three kinds: 1. where such payment is made till the *claim is determined;* 2. where it is made till the *sum becomes due;* 3. where the sum becomes due *at the time* the claim is determined.

Thus, in Prob. XXVII., the sum becoming due at the same time that the claim is determined, the value of the annual payment is obtained by dividing the value of the assurance by unity added to the value of an annuity on the two joint lives AB: that is $p=\frac{A_s}{1+a_{xy}}$.

In Prob. XXVIII. the *claim is determined* on the extinction of the two joint lives; but the sum does not *become due*[1] till the extinction of A's life. Therefore the value of the annual payment till the *claim is determined* will be $p=\frac{A_s}{1+a_{xy}}$; and the value of the annual payment till the sum *becomes due* is $p=\frac{A_s}{1+a_x}$.

In Problem XXIX. the sum becomes due at the time the claim is determined, and consequently the annual payment is equal to the value of the assurance divided by unity added to the value of an annuity on the three joint lives ABC: that is $p=\frac{A_s}{1+a_{xyz}}$.

In Prob. XXX. the claim is not determined, neither does the sum become due, till the extinction of the joint lives AB, and also of the joint lives AC. That is, the annual payment must be made during the continuance of the joint lives AB, and likewise during the continuance of the joint lives AC after the decease of B. The two values will be found equal to $a_{xy}+a_{xz}-a_{xyz}$: consequently we shall have in this case $p=\frac{A_s}{1+a_{xy}+a_{xz}-a_{xyz}}$.

In Prob. XXXI. the annual payment, till the *claim is determined*, will be the same as in the last problem; but the value of such payment till the *sum becomes due* is evidently $p=\frac{A_s}{1+a_x}$.

In a similar manner we might proceed with respect to the remaining problems in Chapter VIII.; but enough has here been said to enable the reader to determine the annual payment in any other case, either of annuities or assurances, that may arise in practice, I therefore shall not detain him with any further remarks on this subject.

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[1] That is, provided the claim is determined *in favour of* the person assuring; and this must be understood in all these cases.

PART SECOND.

PRACTICAL QUESTIONS,

AND

TABLES.

ERRATA.

Page	Line	For	Read
29	26	v^{n-1}	$v^n \times$
35	18	l_{z+m}	$+l_{z+m}$
,,	31	$(n+1)$	$(m+1)$
39	14	denominator	numerator
43	24	$v,$	$v^2,$
60	26, 27	12·263	12·976
63	5, 6	do.	do.
64	22, 23	do.	do.
163	3	$-\frac{v(1+a_{xz\|m})}{2}$	$+\frac{v(1+a_{xz\|m})}{2}$
,,	,,	$v^m l_{xz\|m}$	$v^m l_x l_{z\|m}$
,,	4	$a_{x1 \cdot z}$	$a_{x1 \cdot z\|m}$
167	27	$v(1+a_{xz\|m})$	$\frac{v}{2}(1+a_{xz\|m})$
169	23	$(1-\frac{l_{x1}}{l_z})$	$(1-\frac{l_{z1}}{l_z})$
170	23	$(1-\frac{l_{z1}}{l_z}$	$(1-\frac{l_{z1}}{l_z})$
174	4	$l_{x1 \cdot z}$	$l_{x1} l_z$
,,	9	$v(\mathbf{H}+k)$	$s(\mathbf{H}+k)$
177	3	ecease	decease
180	21	$d_y d_{z\|m}$	$d_{y1} d_{z1\|m}$
,,	,,	d_{y+m}	d_{y1+m}
181	15	$f-v$	$f \times v$

PART SECOND.

CHAPTER XII.

PRACTICAL QUESTIONS TO ILLUSTRATE THE USE OF SOME OF THE PRECEDING PROBLEMS.

QUESTION I.

§ 371. To find the *probability* that a life or lives, of any given age, will continue in being to the end of any given term, according to any given table of observations.

SOLUTION.

In the case of a *single* life, this probability is a fraction whose denominator is the number of persons living at the given age, and whose numerator is the number of persons living at an age older by the given term than the given age.

In the case of *joint* lives it is the product of the probabilities that each of the single lives shall continue in being to the end of the given term. See §§ 23 and 24.

Example 1. The probability that a person, whose age is 20, shall attain to the age of 50, or live thirty years, is, according to the observations of *M. De Parcieux,* as given in Table VII., equal to $\frac{581}{814}$. And the probability that a person, whose age is 40, shall attain to the age of 70, or live thirty years, is, according to the same observations, equal to $\frac{310}{657}$.

But the probability that *both* those persons shall live to the end of thirty years is equal to $\frac{581}{814}$ multiplied by $\frac{310}{657}$; that is, equal to $\frac{180110}{534798}$.

Example 2. The probability that a man aged 46 shall attain to the age of 56, or live ten years, is, according to observations made in *Sweden,* as given in Table VIII., equal to $\frac{3096}{3991}$. And the probability that a woman aged 40 shall attain to the age of 50, or live ten years, is, according to the same observations, equal to $\frac{4027}{4733}$.

But the probability that *both* those persons shall live ten years is equal to $\frac{3096}{3991}$ multiplied by $\frac{4027}{4733}$; that is, equal to $\frac{12467592}{18889403}$.

Example 3. The probability that *each* of three lives, aged 20, 30, and 40, shall live fifteen years, is, according to the observations made at *Northampton*, as given in Table VII., equal to $\frac{4010}{5132}$, $\frac{3248}{4385}$, and $\frac{2448}{3635}$ respectively.

But the probability that *all* those lives shall continue so long is equal to the product of the three fractions into each other: whence such probability will be denoted by $\frac{31883927040}{81801885700}$.

SCHOLIUM.

§ 372. Having thus found the probability that any single or joint lives will continue in being to the end of any given term, we may readily determine the probability that *one or the other* of them will live so long. For, in the case of two lives, the probability here alluded to will be equal to the *difference* between the probability that the joint lives will continue to the end of the term, and the sum of the probabilities that each of the single lives will continue so long.

Thus, in the first example, the probability that one or other of two lives, aged 20 and 40, will continue thirty years, is equal to $\frac{180110}{534798}$ subtracted from $\frac{634057}{534798}$ (or from the sum of the two quantities $\frac{581}{814}$ and $\frac{310}{657}$);[1] which leaves $\frac{453947}{534798}$ for the probability required.

And the probability that one or other of the two lives, mentioned in the second example, will continue ten years, is equal to $\frac{12467592}{18889403}$ subtracted from $\frac{30725125}{18889403}$ (or from the sum of the two quantities $\frac{3096}{3991}$ and $\frac{4027}{4733}$): which leaves $\frac{18257533}{18889403}$ for the probability required.

In like manner, the probability that some one or other, out of three given lives, will continue to the end of any given term, is found by subtracting the sum of the probabilities that each pair of joint lives will continue so long, from the sum of the probabilities that each single life and that the three joint lives will continue the given time, agreeably to the principles laid down in Prob. II.[2]

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[1] These fractions, reduced to a common denominator, are $\frac{381717}{534798}$ and $\frac{252340}{534798}$, the sum of which is equal to $\frac{634057}{534798}$. But it is tedious to operate in this way, and I have adopted it in the present instance for the sake of illustration only. The best method of finding the probabilities, both for single and joint lives, is by means of logarithms; and I would here observe that the logarithm of the denominator subtracted from the logarithm of the numerator will give the logarithm of the probability required, which logarithm will always have a *negative index*.

[2] I shall here mention, by way of note, that the probability that any *two out of three* given lives will continue to the end of any given term, is equal to twice the probability that the three joint lives shall continue the given time, subtracted from the sum of the probabilities that each pair of joint lives shall continue the same period, agreeably to what has been said in the investigation of Prob. III.

QUESTION II.

§ 373. To find the *expectation* of any given life (or lives) receiving a given sum, at the end of any given term.

SOLUTION.

Multiply the present value of the given sum by the probability that the given life (or lives) will continue to the end of the given term, the product will be the answer required. See note to § 42.

Example 1. What is the present value of £1 to be received at the end of thirty years, provided a person, now aged 20, be then alive, interest being reckoned at $4\frac{1}{2}$ per cent., and the probabilities of living as observed by *M. De Parcieux?*

The present value of £1 to be received at the end of thirty years, without any contingency, is by Table III. equal to ·26700; and the probability that a person aged 20 will live thirty years is, by the preceding Question, equal to $\frac{581}{814}$: therefore, these two quantities multiplied together[1] will produce ·1906 for the value required.

In like manner, the expectation of receiving that sum at the end of the same period, provided a person, aged 40, lives so long, is equal to $\frac{310}{657}$ multiplied by ·26700; which produces ·12598 for the value in this case required.

But if the expectation depended on *both* those lives continuing to the end of the term, then $\frac{180110}{534798}$, multiplied by ·26700, will produce ·08992 for the value required.

And if it had depended on *either* of those lives continuing to the end of the term, then $\frac{453947}{534798}$ (or the value found by the Scholium in § 372) being multiplied by ·26700 will produce ·22663 for the value of the expectation in such case required.

Example 2. A man aged 46 will, at the expiration of a lease, which has ten years to run, be entitled to a fine of £1,[2] provided he be then alive: what is his expectation of receiving the same, interest being reckoned at four per cent., and the probabilities of living as observed in *Sweden?*

The present value of £1 certain to be received ten years hence, is, by

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[1] The method of multiplying a vulgar fraction by a decimal fraction, is to multiply the *decimal* by the *numerator* of the vulgar fraction, and to divide the product by the *denominator* of the same.

[2] I have taken the fine equal to *one* pound, because the quantities which result from this assumption will be often referred to in the course of the present chapter; but it is easy to see that the answer here obtained, being multiplied by any other fine, would give the present value of such other fine. Thus, if the fine were £100, the present value of the same, if depending on the life of the man, would be equal to 52·406 or £52, 8s. 1d.; and, if depending on the life of the woman, would be equal to 57·479 or £57, 9s. 7d.

Table III., equal to ·67556; and the probability that a man aged 46 will live ten years is, by the preceding question, equal to $\frac{3096}{3991}$: therefore these two quantities multiplied together will produce ·52406 for the value required.

Had the fine depended on the life of his wife aged 40, then $\frac{4027}{4733}$ multiplied by ·67556, will produce ·57479 for the value in this case required.

But had it depended on their *joint* lives continuing to the end of the given term, then $\frac{12467592}{18889403}$ multiplied by ·67556, will produce ·44589 for the value in such case required.

And had it depended on *either* of those lives continuing so long, then $\frac{18257538}{18889403}$ multiplied by ·67556 will produce ·65296 for the value in this case required.

SCHOLIUM.

§ 374. By means of the general solution here given may be determined all questions relative to the value of such sums as ought to be given for the *Endowments of Children.* Thus, suppose a person has a son aged 11, for whom he wishes to secure £100 on his coming of age; the sum which he ought to pay down for the assurance of the same (reckoning interest at 5 per cent., and the probabilities of living as according to *M. De Parcieux*) is equal to $\frac{806}{872}$ multiplied by 61·391; which produces 56·744, or £56, 14s. 10d. for the answer required.

QUESTION III.

§ 375. To find the value[1] of an annuity on any *Single* life.

SOLUTION.

This value is determined by inspection; for, in either of the Tables which show the values of annuities on any single life, we shall find the value required set down against the age of the given life, according to the several rates of interest at the top of each column.

Example 1. The value of an annuity on a life aged 20, reckoning interest at $4\frac{1}{2}$ per cent., and the probabilities of living as observed by *M. De Parcieux*, is, by Table X., equal to 16·624, or about $16\frac{5}{8}$ years' purchase.[2]

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[1] By the *value* of an annuity I mean the *number of years' purchase* that such an annuity is worth, agreeably to what I have already observed in the note in page 27, and as this mode of expression is used in all the subsequent questions, it will be necessary to bear this observation in mind.

[2] The *number of years' purchase* being multiplied by the annuity will give the total present value of the same. Thus, if the annuity in the present instance were £50 per annum: then 16·624 multiplied by 50 would give 831·200, or £831, 4s. for the value of the same.

Had the life been 40 years of age, the value would have been equal to 14·254. Or had the rate of interest been in each case 5 per cent., the values would have been equal to 15·469 and 13·459 respectively.

Example 2. The value of an annuity on the life of a man aged 46, reckoning interest at 4 per cent. and the probabilities of living as observed in *Sweden* is, by Table X., equal to 12·297, or rather more than 12¼ years' purchase.

Had the annuity been on the life of a *woman* aged 40, the value would have been equal to 14·401. Or had the rate of interest in each case been 5 per cent., the values would have been 11·153 and 12·856 respectively.

QUESTION IV.

§ 376. To find the value of an annuity on *two Joint Lives.*

SOLUTION.

Look in the Tables which show the values of annuities on two joint lives of all ages; and if the two lives have the same common age, or if their difference of age comes within the limits of those tables, the value of an annuity on their joint continuance will be found expressed therein.

Example 1. The value of an annuity on two joint lives aged 20 and 40, interest being reckoned at 4½ per cent., and the probabilities of living as observed by *M. De Parcieux*, is, by Table XI., equal to 12·545; or rather more than 12½ years' purchase.

Had both the lives been 20 years of age, the value would have been, by the same Table, equal to 14·004; or, had they been both 40 years of age, the value would have been 11·710.

Example 2. The value of an annuity on the joint lives of a man aged 46 and his wife aged 40, reckoning interest at 4 per cent. and the

Or, if the annuity had been £4, 10s. per annum; then 16·624 multiplied by 4·5 would give 74·808, or £74, 16s. 2d. for the value in this case required. This method is universal, and applies to all cases of annuities, whether present or in reversion, whether temporary or deferred; and therefore it will be sufficient, in all the subsequent examples, to deduce the value of an annuity of *one* pound per annum; or, in other words, to find the *number of years' purchase.*

Having thus found the number of years' purchase that ought to be given for an annuity, we may readily determine the *annuity* that ought to be given for any given *sum* invested, merely by dividing such sum by the number of years' purchase. Thus, if a person wished to lay out £4000 in the purchase of such an annuity as the one mentioned in the text, the annuity which he ought to receive for that money will be found by dividing 4000 by 16·624: whence 240·616, or £240, 12s. 4d., will be the annuity required. This method is likewise universal, and therefore it will be unnecessary to repeat it in any of the subsequent cases. The same principles will apply to the value of reversionary sums, for which, see Question XXVII.

probabilities of living as observed in *Sweden*, is, by Table XII., equal to 10·286.

Had both the lives been 40 years old the value would have been, by the same Table, equal to 10·964; or, had they been both 46 years old, the value would have been 9·736.[1]

SCHOLIUM.

§ 377. If the *difference* of age between the two lives is any number of years not given in the tables, the required value may be easily obtained by means of the following rule:—

Find, by the tables, the value of an annuity on two joint lives whose difference of age is *greater* than, but at the same time *nearest* to, the difference of age between the proposed lives, and the oldest of which is of the same age with the oldest of the proposed lives. Find also, by the same tables, the value of an annuity on two joint lives whose difference of age is the *next less* to that just mentioned; and the oldest of which is, in like manner, of the same age with the oldest of the proposed lives. Then will the 1st, 2d, 3d, &c., arithmetical *mean*[2] between the least and the greatest of these two values be the value required, according as one of the proposed lives is 1, 2, 3, &c., years younger than the other.

Example 1. Let it be required to find the value of an annuity on two joint lives aged 32 and 50; at the rate of 4½ per cent. interest, and according to the probabilities of life as observed by *M. De Parcieux*?

That difference of age which is *greater* than the difference between these lives, but at the same time *nearest* to it, is 20; and the value of an annuity on two joint lives whose difference of age is twenty years, and the oldest of which is of the same age with the oldest of the proposed lives (that is, the value of an annuity on two joint lives aged 30 and 50) is, by Table XI., equal to 10·611. And the value of an annuity on two joint lives whose difference of age is next less to 20 (that is, whose difference of age is 10 years) and the oldest of which is of the same age with the oldest of the proposed lives (that is, the value of an annuity on two joint lives aged 40 and 50) is, by the same Table, equal to 10·274. Therefore, these being the values of an annuity on two joint lives aged 30 and 50, and on two joint lives aged 40 and 50, it is evident that the value of an annuity on two joint lives, aged 32 and 50, will be nearly equal to the least of these two values increased by 8-*tenths* of the difference between them; or (which is the same thing) equal to the greatest value diminished by 2 tenths of their difference. Now, the difference between these values is equal to ·337; *one*-tenth of which is equal to ·0337, and *two*-tenths are therefore equal to ·067. Consequently 10·611, diminished by ·067, will leave 10·544 for the value required of an annuity on the two joint lives aged 32 and 50.

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[1] The values of annuities on the joint lives in Table XII. are deduced from the probabilities of living amongst males and females *collectively*, and therefore do not show the true values of annuities on two joint lives, one of which is a *male* and the other a *female*. Tables formed upon this latter principle are still a *desideratum*. See the example in § 34.

[2] The tables for the values of annuities on two joint lives, according to the *Northampton* observations, are the only ones where the difference of age is so small as *five* years. In the tables deduced from the observations in *Sweden*, the difference of age is *six* years; and in those deduced from the observations of *M. De Parcieux*, the difference of age is *ten* years. Consequently the 1st, 2d, 3d, &c., arithmetical mean between the least and greatest of any two values, according to the *Northampton* tables, will be equal to the least value increased by 1, 2, 3, &c., *fifths* of their difference; but according to the *Swedish* tables it will be equal to the least value increased by 1, 2, 3, &c., *sixths* of their difference; and according to the tables of *M. De Parcieux*, it will be equal to the least value increased by 1, 2, 3, &c., *tenths* of their difference.

Example 2. Let it be required to find the value of an annuity on two joint lives aged 20 and 60; at the rate of 4 per cent. interest, and according to the probabilities of living as observed in *Sweden*.

The difference of age which is greater than the difference between these lives, but at the same time nearest to it, is 42; and the value of an annuity on two joint lives whose difference of age is 42 years, and the oldest of which is of the same age with the oldest of the proposed lives (that is, the value of an annuity on two joint lives aged 18 and 60) is, by Table XII., equal to 8·208. And the value of an annuity on two joint lives whose difference of age is 6 years less than 40, and the oldest of which is likewise of the same age with the oldest of the proposed lives (that is, the value of an annuity on two joint lives aged 24 and 60) is, by the same Table, equal to 8·097. Therefore, these being values of an annuity on two joint lives aged 18 and 60, and on two joint lives aged 24 and 60, it follows that the value of an annuity on the two joint lives 20 and 60 will be nearly equal to the least of these two values increased by 4-*sixths* of the difference between them. Now, their difference being equal to ·111, it follows that *one*-sixth of such difference will be ·0185; and *four*-sixths of such difference will be ·074: which being added to 8·097 will give 8·171 for the required value of an annuity on the two joint lives aged 20 and 60.

Example 3. What is the value of an annuity on two joint lives aged 26 and 60; reckoning interest at 4 per cent., and probabilities of life as observed at *Northampton*?

The difference of age which is greater than the difference between these two lives, but at the same time nearest to it, is 35; and the value of an annuity on two joint lives whose difference of age is 35, and the oldest of which is equal to the oldest of the proposed lives (that is, the value of an annuity on two joint lives aged 25 and 60) is, by Table XIII., equal to 7·906. And the value of an annuity on two joint lives whose difference of age is five years less than 35, and the oldest of which is also of the same age with the oldest of the proposed lives (that is, the value of an annuity on two joint lives aged 30 and 60) is, by the same Table, equal to 7·802. Therefore, these being the values of an annuity on two joint lives aged 25 and 60, and on two joint lives aged 30 and 60, it follows that the value of an annuity on the two joint lives 26 and 60 will be nearly equal to the least of these values increased by 4-*fifths* of the difference between them, or nearly equal to the greatest of these values decreased by *one-fifth* of their difference. Now, this difference being ·104, it is evident that *one-fifth* of it is equal to ·021; which being deducted from 7·906,[1] will give 7·885 for the value required of an annuity on the two joint lives 26 and 60.

§ 378. Since the tables of the values of annuities on two joint lives, according to the observations of *M. De Parcieux*, are calculated only for such lives whose difference of age is *ten* years, it is evident that the method just laid down (for determining the values of annuities on two joint lives, whose difference of age is any intermediate number) will not be quite so correct as from those tables calculated according to the observations of life in *Sweden*, where the difference of age is *six* years. Neither will these latter ones show the value, for such intermediate ages, so correctly as the tables calculated according to the observations of life at *Northampton*, where the difference of age is *five* years. In neither case will the error be very considerable; but in the latter case particularly (where the tables show the values of annuities on two joint lives of all ages whose difference is not more than 5 years) the error is so trifling as to be not worth considering. This will evidently appear from the following comparison (given by Dr. Price in his *Observations on Reversionary Payments*, vol. ii. page 359) of the values of annuities on two joint lives of the ages therein mentioned, deduced from the *Northampton* observations, interest at 3 per cent.:—

Ages.	Value by Rule.	Correct Value.
18—14	14·972	14·978
18—15	14·858	14·864
18—16	14·744	14·744
18—17	14·630	14·626
45—31	10·862	10·869
45—32	10·802	10·811
45—33	10·742	10·751
45—34	10·682	10·688
66—27	7·092	7·095
66—28	7·076	7·080
66—29	7·060	7·063
66—30	7·044	7·046

In the higher rates of interest the agreement is greater.

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[1] Or we may add four-fifths to the least value, which would give the same result.

Dr. Price was enabled to make this comparison by the tables in the office of the *Equitable Society;* where, in order to lay the foundation of accuracy in conducting the business of the office, it has been thought necessary to compute minutely to four places of decimals the values by the *Northampton* observations, at 3 per cent., of two joint lives, *for every possible difference of age.*

§ 379. When one of the given lives is under 10 years of age, we ought, in deducing the values agreeably to this rule, to attend particularly to the order of the difference between the values taken from the Tables; that is, to observe whether such difference is *in*creasing or *de*creasing. For instance, suppose it is required to determine the value of an annuity on two joint lives aged 9 and 30, interest at 3 per cent., and the probabilities of living as at *Northampton:* the rule directs us to find the value of an annuity on two joint lives aged 5 and 30, and on two joint lives 10 and 30, which are respectively equal to 13·762 and 14·150; and that ·078 (or one-fifth of the difference between them) being subtracted from the latter value, will give 14·072 for the value of an annuity on two joint lives aged 9 and 30. But the following comparison will show this to be incorrect: for if we take out the values of annuities on the several joint lives as under, viz.:—

5—30 = 13·762
10—30 = 14·150
15—30 = 13·734
20—30 = 13·286
25—30 = 12·966
30—30 = 12·589

it will be seen that (beginning at the bottom) the values gradually *in*crease till we come to the age of 10 and 30; and therefore that the value of an annuity on any two joint lives, one of which is 30 years of age, and the other of any age between 10 and 30, will be deduced accurately enough by means of the rule above given. And this also would be the case with respect to the value of annuities on any two joint lives, one of which is 30 years and the other of any age *below* 10 years of age, provided the *de*crease commenced exactly at the age of 10 years; but it is probable that the decrease does not begin to take place till about the joint ages of 8 and 30;[1] and consequently that the value of an annuity on two joint lives aged 9 and 30 is *greater* than 14·150, instead of being *less.* The proper method, therefore, of finding the value of an annuity on the two joint lives aged 9 and 30 will be to take ·083 (or one-fifth of the difference between 14·150 and 13·734) and *add* it to 14·150: which will give 14·233 for the value of an annuity on the two joint lives aged 9 and 30. These cases have never yet been noticed by any preceding writer, although they frequently occur in practice.

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[1] The period at which this decrease commences, varies according to the rate of interest and according to the difference between the ages of the two lives.

QUESTION V.

§ 380. To find the value of an annuity on *three Joint* lives.

SOLUTION.

Look in Table XIV.; and if the three lives have the same common age, or if their difference of ages be 10 and 20 years, the value of an annuity on their joint continuance will be found expressed therein.

Example. The value of an annuity on three joint lives aged 20, 30, and 40, reckoning interest at 4 per cent., and the probabilities of living as at *Northampton*, is equal to 8·986: but had all the lives been 20 years of age, the value would have been equal to 10·342; or had they all been 40 years of age, the value would have been equal to 7·865.

SCHOLIUM.

§ 381. It unfortunately happens that the two tables above mentioned are the only ones that have been published for determining the values of annuities on three joint lives. The labour of computing such tables is so very great, and the combinations of ages are so various, that it will probably be a long time before any person will undertake to finish what has been here begun: and till that is the case we may make use of the following general and very easy rule, given by Mr. Simpson, for finding the values of annuities on any *three*, from the values of any *two*, joint lives:—

"Let A be the youngest, and C the oldest of the three proposed lives. Take the value of an annuity on the two joint lives B and C, and find the age of a single life D of the same value. Then find the value of an annuity on the two joint lives A and D, which will be the value required."

Example. What is the value of an annuity on three joint lives aged 10, 20, and 30; interest at 4 per cent., and the probabilities of living as at *Northampton*?[1]

The value of an annuity on the two joint lives aged 20 and 30 is, by Table XIII., equal to 11·873; which, being compared with the values in Table X., will be found equal to the value of an annuity on a single life

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[1] It will readily appear that we can obtain the values of annuities on *three* joint lives more correctly from the *Northampton* tables of *two* joint lives, than from any other observations; because they are as yet the most comprehensive, and include the greatest variety of combined ages.

D aged $47\frac{17}{205}$,[1] or 47 years and 1 month. And the value of an annuity on the joint lives A and D (that is, on two joint lives aged 10 and $47\frac{17}{205}$) is, by the rule in the preceding scholium, equal to 10·474;[2] which is the value required.

Had the two oldest lives been both 40 years of age, and the youngest 20, the value of an annuity on the joint lives of the two former would, by Table XIII., be equal to 9·820, answering to a single life D aged $56\frac{157}{228}$. And the value of an annuity on the joint lives A and D (that is, on two joint lives aged 20 and $56\frac{157}{228}$) is, by the rule alluded to in the last note, equal to 8·601: which is the value required of an annuity on three joint lives aged 20, 40, and 40.

Or had the two youngest lives been 20, and the eldest 40 years of age, then the value of an annuity on two joint lives aged 20 and 40 would, by Table XII., be equal to 10·924; answering to a single life D aged $51\frac{133}{208}$. And the value of an annuity on the two joint lives A and D (that is, on two joint lives aged 20 and $51\frac{133}{208}$) is, by the rule alluded to in the preceding note, equal to 9·406; which is the value required of an annuity on three joint lives aged 20, 20, and 40.

The following table (computed from the probabilities of life as observed at *Northampton*, and reckoning interest at 4 per cent.) will show how nearly the rule, above explained, approximates to the true values as given in Table XIV.:—

Ages.	Value by Rule.	Correct Value.	Ages.	Value by Rule.	Correct Value.
10—20—30	10·474	10·438	20—20—20	10·516	10·342
15—25—35	9·836	9·738	25—25—25	9·937	9·796
20—30—40	9·097	8·986	30—30—30	9·351	9·221
25—35—45	8·390	8·313	35—35—35	8·703	8·585
30—40—50	7·651	7·571	40—40—40	7·983	7·865
35—45—55	6·884	6·816	45—45—45	7·243	7·126
40—50—60	6·046	5·994	50—50—50	6·433	6·317
45—55—65	5·175	5·145	55—55—55	5·637	5·550
50—60—70	4·235	4·219	60—60—60	4·817	4·755
55—65—75	3·308	3·298	65—65—65	3·936	3·914
10—10—10	12·206	12·200	70—70—70	3·010	2·995
15—15—15	11·376	11·274	75—75—75	2·118	2·119

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[1] The value, in Table X., which is next greater than 11·873, is 11·890; which is the value of an annuity on a single life aged 47. The difference between these values, or 17, is the *numerator* of the fraction: and the *denominator* is the difference between 11·685 (or the next less value to 11·873) and 11·890.

[2] The value of an annuity on two joint lives aged 10 and 47 is, by the rule in the preceding scholium, equal to 10·485: and the value of an annuity on two joint lives aged 10 and 48 is, by the same rule, equal to 10·356. The difference between these two values, or ·129, being multiplied by $\frac{17}{205}$, will give ·011; which being subtracted from 10·485 will leave 10·474 for the value required. This shows the true method of proceeding in such cases; but if this fraction be either very small, or does not differ much from unity, the error will not be considerable, if (for the sake of more expedition) D is always taken for that age, whether greater or less, which answers most nearly to the value of the annuity on the joint lives B and C, without regarding the fraction.

From which it may be inferred that this rule will give the values of annuities on three joint lives generally within a *ninth* or a *tenth*, and sometimes within less than a *twentieth* part of a year's purchase. It may also be observed that when the oldest of the three ages does not exceed 75, and the youngest is not less than 10, the error falls on the side of *excess*; and, consequently, that if ·05 (or the twentieth part of a year's purchase) be deducted from the values by the rule, we shall obtain the true value, in some cases almost exactly, and in most cases, much more nearly.

QUESTION VI.

§ 382. To find the value of a *Deferred* annuity on any *single* or *joint lives.*[1]

SOLUTION.

Find the value of an annuity on a life, or joint lives, as many years older than the given life, or joint lives, as are equal to the term during which the annuity is deferred; find also the expectation of the given life, or joint lives, receiving £1 at the end of that term: the product of these two quantities will be the answer required. See § 45.

Example 1. A person aged 20 wants to purchase an annuity for what may happen to remain of his life after the term of 30 years: what is the present value of the same, reckoning interest at 4½ per cent., and the probabilities of life as observed by *M. De Parcieux?*

The value of an annuity on a life aged 50, is, by Table X., equal to 11·921; and the expectation of a life aged 20 receiving £1 at the end of thirty years is, by Question II., equal to ·1906: therefore 11·921 multiplied by ·1906 will produce 2·272 for the number of years' purchase required.

Had the life been 40 years of age, the value would have been equal to 6·221 multiplied by ·1260; which would produce ·784 for the value required.

Example 2. A man now aged 46 will at the end of ten years come into possession of an annuity on his own life: what is the present value of the same, reckoning interest at 4 per cent., and the probabilities of living as observed in *Sweden?*

The value of an annuity on a male aged 56 is, by Table X., equal to 9·717; and the expectation that a man aged 46 will receive £1 at the end of ten years is, by Question II., equal to ·5241: therefore these two quantities being multiplied together will give 5·093 for the value required.

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[1] This Question is of considerable utility in enabling us to determine the best means of providing Annuities for the *benefit of old age,* as will be more fully explained in the following chapter.

Had the annuitant been a female aged 40, then 12·049 multiplied by ·5748 would give 6·926 for the value in this case required.

Example 3. Two persons aged 20 and 40 wish to purchase an annuity for the remainder of their joint lives after thirty years: what ought they to give for the same, reckoning interest at 4½ per cent., and the probabilities of living as observed by *M. De Parcieux?*

The value of an annuity on two joint lives aged 50 and 70 is, by Table XI., equal to 5·517; and the expectation of two joint lives, aged 20 and 40, receiving £1 at the end of thirty years is, by Question II., equal to ·0899: therefore the product of these two quantities will give ·496 for the value required.

Example 4. A man aged 46, together with his wife aged 40, are entitled to an annuity on their joint lives, to commence at the end of ten years: what is the value of their interest therein, taking the probabilities of life as observed in *Sweden*, and the rate of interest at 4 per cent.?

The value of an annuity on the joint lives of two persons, a man aged 56 and a woman aged 50, is, by Table XII., equal to 7·874; which being multiplied by ·4459 (or the value of the expectation of two joint lives aged 46 and 40, receiving £1 at the end of ten years, as found by Question II.) will produce 3·511 for the value required.

SCHOLIUM.

§ 383. If, instead of determining the value of a deferred annuity in a *single* payment, we wish to determine the value of the same in *annual* payments during the term for which the annuity is deferred;[1] the amount of those annual payments is readily obtained by means of the following rule:—

Divide the value of the annuity in a *single* payment, by unity added to the value of a similar temporary annuity for one year less than the given term: the quotient will be the annual payment required. See § 366.

Example 1. A person aged 20 wants to purchase an annuity for what may happen to remain of his life after the term of thirty years: what sum ought he to give *annually* to the end of that term[2] in order to have the same assured to him, reckoning interest at 4½ per cent., and the probabilities of living as observed by *M. De Parcieux?*

The value of this deferred annuity in a *single* payment is, by the first example to the Question, equal to 2·272; and the value of a similar temporary annuity for twenty-nine years is, by the rule in the following Question,[1] equal to 14·161; therefore, 2·272 divided by 15·161 will give ·150 for the value of the annual payments during the term deferred.

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[1] The first of those annual payments to be made *immediately*, and the remaining ones at the *beginning* of every subsequent year; since this is the usual method of making such annual payments.

[2] Such annual payments, however, subject to failure, in case the given life becomes extinct before the end of that term.

In like manner we might determine the value in annual payments of an annuity on the life of a woman for what may happen to remain of it after ten years, reckoning interest at 4 per cent., and the probabilities of living as observed in *Sweden*.

For the value of this deferred annuity in a *single* payment is, by the second example to the Question, equal to 6·926; and the value of a similar temporary annuity for nine years is, by the rule just alluded to, equal to 6·900: therefore, 6·926 divided by 7·900 will give ·877 for the value of the annual payments required.

Example 2. A man aged 46 and his wife aged 40 are entitled to an annuity on their joint lives, to commence at the end of ten years, but are willing to surrender their interest in the same for an equivalent annuity (commencing immediately) during such term: what ought that equivalent annuity to be, reckoning interest at 4 per cent., and the probabilities of living as observed in *Sweden?*

The value of the deferred annuity on the joint lives is, by the fourth example to the Question, equal to 3·511; and the value of a similar temporary annuity for nine years is, by the following Question (or the rule in the preceding note), equal to 6·329; therefore 3·511 divided by 7·329 will give ·479 for the value of the annual payments during the term deferred.

QUESTION VII.

§ 384. To find the value of a *Temporary* annuity on any *single* or *joint* lives.[2]

SOLUTION.

From the value of an annuity on the given single or joint lives, deduct the value of an annuity on the same lives *deferred* during the given term: the remainder will be the value required. See § 47.

Example 1. A person aged 20 buys an annuity for thirty years, on condition that if he dies before the expiration of that term the annuity shall cease: what ought he to give for the same, reckoning interest at 4½ per cent., and the probabilities of living as observed by *M. De Parcieux?*

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[1] A more convenient method however of determining *such* temporary annuities is expressed by the following rule:—To the value of the deferred annuity add the expectation that the given life or lives shall receive £1 at the end of the given term; subtract the sum from the value of an annuity on the given life or lives; the difference will be the value of the temporary annuity for *one year less than the given term.* See note 2 in page 206.

[2] I call a *temporary* annuity, one that is to continue during a given *term* only; which term is *less* than that to which it is possible the life or lives may extend. See note 1 in page 36.

The value of an annuity on a life aged 20 is, by Table X., equal to 16·624; and the value of an annuity on the same life, deferred for thirty years, is, by Question VI., equal to 2·272: consequently this value, subtracted from the former, will leave 14·352 for the answer required.

Had the life been 40 years of age, then ·784 (or the value of an annuity on such life deferred for thirty years, as found by Question VI.), deducted from 14·254, would leave 13·470 for the value in this case required.

Or, had these two persons (aged 20 and 40) purchased the annuity on their *joint* lives, then ·496 (or the value of an annuity on such joint lives deferred for thirty years, as found by Question VI.), being deducted from 12·545, will leave 12·049 for the value in this case required.

Example 2. A man aged 46 is entitled to the rent of an estate for ten years, provided he lives so long: what is the value of his interest therein, reckoning interest at 4 per cent., and the probabilities of living as observed in *Sweden?*

The value of an annuity on such life is, by Table X., equal to 12·297; and the value of an annuity on the same life, deferred for ten years, is, by Question VI., equal to 5·093; consequently the difference between these two values, or 7·204, will be the value required.

Had the estate depended on the life of his wife aged 40; then 7·475 (or the difference between 14·401 and 6·926) would be the value of the temporary annuity in this case required.

Or had the estate depended on their *joint* lives, then 6·775 (or the difference between 10·286 and 3·511) would be the value of the temporary annuity in this case required.

QUESTION VIII.

§ 385. To find the value of an annuity on the *Longest* of *two* lives.

SOLUTION.

From the sum of the values of an annuity on the two single lives, subtract the value of an annuity on the two joint lives, the difference will be the value required. See § 56.

Example 1. What is the value of an annuity on the longest of two lives aged 20 and 40; interest at 4½ per cent., and the probabilities of living as observed by *M. De Parcieux?*

The value of an annuity on the two single lives is, by Table X., equal to 16·624 and 14·254 respectively, the sum of which is 30·878; therefore, if from this we subtract 12·545, or the value of an annuity on the two joint lives as found by Table XI., the difference, or 18·333, will be the value required.

Had the ages of the given lives been 50 and 70, the sum of the values of an annuity on their single lives would, by Table X., be equal to 18·142 (that is, equal to 11·921 added to 6·221); and the value of an annuity on their joint lives would, by Table XI., be equal to 5·517; consequently 12·625, or the difference between these two values, would be the value of an annuity on the longest of their two lives.

Had both the lives been 20 years of age, the value of an annuity on their single lives would (according to the same rate of interest, &c., have been equal to twice 16·624; that is, equal to 33·248: and the value of an annuity on their joint lives would be equal to 14·004: therefore the difference between these two values, or 19·244, would be the number of years' purchase in this case required.

Example 2. What is the value of an annuity on the longest of two lives, a man and his wife, the former aged 46 and the latter aged 40; interest at 4 per cent., and the probabilities of living as observed in *Sweden?*

The value of an annuity on the life of the man is, by Table X., equal to 12·297, and the value of an annuity on the life of the woman is 14·401; the sum of these is 26·698, from which we must subtract 10·286, the value of an annuity on their joint lives by Table XII.; and the difference, or 16·412, will be the value of an annuity on the longest of their two lives.

Had the two lives been each of them ten years older, or 56 and 50 years of age, then the sum of the values of an annuity on their single lives would, by Table X., be equal to 21·766 (that is, equal to 9·717 added to 12·049), and the value of an annuity on their joint lives would, by Table XII., be equal to 7·874; consequently 13·892, or the difference between these two values, would be the value of an annuity on the longest of their lives.

Had both the lives been 40 years of age, then 10·964 (or the value of an annuity on their joint lives) subtracted from 28·069 (or the sum of the values of an annuity on their single lives[1] would give 17·105 for the answer in this case required.

QUESTION IX.

§ 386. To find the value of an annuity on the *Longest* of *three* lives.

SOLUTION.

From the sum of the values of an annuity on all the single lives, subtract the sum of the values of an annuity on each *pair* of joint lives, and to the difference add the value of an annuity on the three joint lives: this last sum will be the value required. See § 56.

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[1] The value of an annuity on the life of the man is 13·668, and the value of an annuity on the life of the woman is 14·401.

Example. What is the value of an annuity on the longest of three lives, aged 20, 30, and 40; interest at 4 per cent., and the probabilities of living as at *Northampton?*

The value of an annuity on each single life is, by Table X., equal to 16·033, 14·781, and 13·197 respectively, the sum of which is 44·011; the value of an annuity on each pair of joint lives (viz., 20 and 30, 20 and 40, 30 and 40) is, by Table XIII., equal to 11·873, 10·924, and 10·490 respectively, the sum of which is 33·287; the difference between these two values is 10·724, which being added to 8·986 (or the value of an annuity on the three joint lives, as found by Table XIV.), will give 19·710 for the number of years' purchase required.

Had all three lives been 20 years of age, the value of an annuity on their single lives would have been equal to thrice 16·033, or 48·099; the value of an annuity on each pair of joint lives would have been equal to 37·605, or to thrice 12·535 (that is, equal to thrice the value of an annuity on two joint lives both aged 20, as found by Table XIII.); and the value of an annuity on the three joint lives would, by Table XIV., be equal to 10·342: therefore 20·836 would be the number of years' purchase in this case required.

QUESTION X.

§ 387. To find the value of an annuity granted upon *three* lives, but to continue only as long as any *two* of them are in being together.

SOLUTION.

From the sum of the values of an annuity on each *pair* of joint lives, subtract twice the value of an annuity on the three joint lives, the difference will be the value required. See § 64.

Example. An annuity is purchased upon three lives aged 20, 30, and 40; on this condition, that as soon as any two of the lives fail, the annuity shall cease: the value of the same is required, reckoning interest at 4 per cent., and the probabilities of living as at *Northampton?*

The value of an annuity on each pair of joint lives (viz., 20 and 30, 20 and 40, 30 and 40) is, by Table XIII., equal to 11·873, 10·924, and 10·490 respectively, the sum of which is 33·287; and the value of an annuity on the three joint lives is 8·986: therefore twice the latter quantity, or 17·972, subtracted from 33·287, will give 15·315 for the number of years' purchase required.

Had the ages of all the three lives been 20 years, the value would, in this case, have come out equal to 16·921.

QUESTION XI.

§ 388. To find the value of an annuity, on the *longest* of any number of lives, *Deferred* for any given term.

SOLUTION.

Substitute the values of *deferred* annuities on each single and joint lives, instead of the annuities for the *whole continuance* of those lives, and proceed as in the solutions to the two preceding questions. See § 60.

Example 1. What is the value of an annuity granted on the longest of two lives aged 20 and 40, but which is not to be entered on or enjoyed till after the expiration of thirty years; reckoning interest at 4½ per cent., and the probabilities of living as observed by *M. De Parcieux?*

The value of a deferred annuity for thirty years on a life aged 20 is, by Question VI., equal to 2·272; the value of a similar annuity on a life of 40 is equal to ·784; and the value of a similar annuity on the two joint lives is equal to ·496: therefore, if from the sum of the two former, or 3·056, we subtract the latter, the difference, or 2·560, will be the value required.

Example 2. A man and his wife (the former aged 46, and the latter aged 40) purchase on the longest of their two lives the reversion of the lease of an estate, which they are not to enter upon till the end of ten years: what is the present value of the same, interest being reckoned at 4 per cent., and the probabilities of living as observed in *Sweden?*

The value of an annuity on the life of a male aged 46, deferred ten years, is, by Question VI., equal to 5·093; the value of a similar annuity on a female aged 40, is, by the same question, equal to 6·926; and the value of a similar annuity on their joint lives is equal to 3·511. Consequently this latter value deducted from the sum of the two former ones will leave 8·508 for the answer required.

§ 389. These examples give the present values in a *single* payment; but, if we wish to determine the same value in *annual* payments commencing immediately, we must divide the single payment thus found, by unity added to the value of an annuity on the longest of the given lives for one year less than the given term.

Thus in the second example, the value of the deferred annuity in a *single* payment is 8·508; and, by the rule in the following question,[1] the value of an annuity on the longest of the two lives for nine years is equal to 7·251: consequently 8·508 divided by 8·251 will give 1·031 for the value in *annual* payments.

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[1] A more convenient method, however, of determining *such* temporary annuities is expressed by the following rule. To the value of the deferred annuity on the longest of the given lives add the expectation that the *longest* of such lives shall receive £1 at the end of the given term; subtract the sum from the value of an annuity on the longest of the given lives: the difference will be the value of the annuity for *one year less than the given term*. See note 2 in p. 206.

SCHOLIUM.

§ 390. It should here be particularly observed that if the deferred annuity, mentioned in this question, depends upon the *joint* existence of all the lives, to the end of the given term, the solution will be materially different; and these two cases must not be confounded. In this latter case, the value will be equal to the value of an annuity on the longest of the same number of lives (each older by the given term than the given lives) multiplied by the expectation that the *joint* lives shall receive £1 at the end of that term. See § 61.

Example 1. What is the value of an annuity on the longest of two lives aged 20 and 40, but which is not to be entered upon till the end of thirty years, and then only in case *both* the lives are in existence; interest at 4½ per cent., and the probabilities of life as observed by *M. De Parcieux?*

The value of an annuity on the longest of two lives aged 50 and 70 is, by the rule in Question VIII., equal to 12·625; and the expectation, that two lives aged 20 and 40 will receive £1 at the end of thirty years, is, by Question II., equal to ·0899 : the product of these two quantities will give 1·135 for the answer required.

Example 2. A man (aged 46) and his wife (aged 40) purchase an annuity on the longest of their two lives, which is to commence at the end of ten years, provided they are *both* alive: what is the present value of the same, interest at 4 per cent., and the probabilities of life as observed in *Sweden?*

The value of an annuity on the longest of two lives (a man aged 56, and a woman aged 50) is, by the rule in Question VIII., equal to 13·892; and the expectation that two such lives aged 46 and 40 will receive £1 at the end of ten years is, by Question II., equal to ·4459: the product of these two quantities will give 6·194 for the value required.

§ 391. The value of these annuities in *annual* payments commencing immediately will be equal to the value in a *single* payment, divided by unity added to the value of an annuity on the *joint* lives for one year less than the given term.

Thus, in the second example, the value of the deferred annuity in a *single* payment is equal to 6·194: and by the rule in note 1, page 224, the value of an annuity on the two joint lives deferred for nine years is 6·329 : consequently 6·194 divided by 7·329 will give ·845 for the value in annual payments.

QUESTION XII.

§ 392. To find the value of a *Temporary* annuity on the *longest* of any number of lives.

SOLUTION.

From the absolute value of an annuity on the longest of the given lives, subtract the value of the same annuity *deferred* during the given term: the difference will be the value required.[1] See § 62.

Example 1. What is the value of a temporary annuity for thirty years on the longest of two lives aged 20 and 40, reckoning interest at 4½ per cent., and the probabilities of life as given by *M. De Parcieux?*

The value of an annuity on the longest of those lives is, by Question VIII., equal to 18·333; and the value of an annuity on the longest of those lives *deferred* for thirty years is, by Question XI., equal to 2·560; consequently the difference between these two values, or 15·773, will be the answer required.

Example 2. A man aged 46 purchases an annuity for ten years, terminable, however, at any time prior thereto, on the extinction of his own life and the life of his wife aged 40: what is the value of the same, interest at 4 per cent., and the probabilities of living as observed in *Sweden?*

The value of an annuity on the longest of their lives is, by Question VIII., equal to 16·412; and the value of an annuity on the longest of their lives *deferred* for ten years is, by Question XI., equal to 8·508: the difference therefore between these two values, or 7·904, is the answer required.

QUESTION XIII.

§ 393. To find the value of the *Reversion* of an annuity on a *single* life after any other single life.[2]

SOLUTION.

From the value of an annuity on the life in reversion, subtract the value of an annuity on the two joint lives; the difference will be the value required. See § 76.

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[1] Or (which is the same thing), substitute the values of *temporary* annuities on each single and joint lives, instead of the values of annuities for the *whole continuance* of those lives; and proceed as in the solutions to Questions VIII. and IX.

[2] This Question, and also Question XVIII., are of considerable importance in enabling us to determine the best means of providing Annuities for the *benefit of Widows;* as will be more fully explained in the following Chapter.

Example 1. A person aged 20 wishes to purchase an annuity for what may happen to remain of his life beyond another life aged 40: what ought he to give for the same, allowing interest at 4½ per cent., and the probabilities of living as observed by *M. De Parcieux?*

The value of an annuity on the life in reversion (that is, on the life aged 20) is, by Table X., equal to 16·624; and the value of an annuity on the two joint lives is, by Table XI., equal to 12·545; therefore the difference of these two values, or 4·079, is the number of years' purchase required.

Had the life in reversion been 40, and the life in possession 20 years of age, the value would have come out equal to 1·709.

Or, had both the lives been 20 years, the value would have come out equal to 2·620: and had they both been 40 years of age, the value would have come out equal to 2·544.

Example 2. What is the value of an annuity to be enjoyed by a woman aged 40, during her life, after the decease of her husband aged 46; interest at 4 per cent., and the probabilities of living as amongst males and females respectively in *Sweden*.[1]

The value of an annuity on the life of a woman aged 40 is, by Table X., equal to 14·401, and the value of an annuity on their joint lives is, by Table XII., equal to 10·286; therefore 4·115 is the number of years' purchase required.

Had their lives been both 40 years of age, the value would have come out equal to 3·437.[1]

SCHOLIUM.

§ 394. If, instead of a *single* payment, we wish to determine the value of these reversionary annuities in *annual* payments to be made during the existence of the two joint lives, we must divide the value, found in either case, by unity added to the value of an annuity on the joint lives; and the quotient will give the annual payments required.

Thus, in the first example, 4·079 being divided by 13·545 will give ·301 for the *annual* payments which ought to be made during the joint lives, as an equivalent for the sum in a single payment. In like manner, in the second example, 4·115 being divided by 11·286 will give ·365 for the *annual* payments which ought to be made by a man aged 46 during the joint lives of himself and his wife aged 40, in order to secure to his widow, on his death, an annuity of £1 per annum during her life.[1]

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[1] It is worthy of remark that the value of a reversionary annuity on one life after another is, when the difference of age is not very considerable, *less* in the younger ages and *greatest* in the middle ages of life: a circumstance which may be attributed to the higher chances of living in the younger ages, whereby the probability of *survivorship* is deferred so long as to affect in a material degree the value of the *reversionary* annuity.

QUESTION XIV.

§ 395. To find the value of the *Reversion* of an annuity on a *single* life A after the *longest* of two other lives, P and Q.

SOLUTION.

From the sum of the values of an annuity on the single life A in reversion, and on the three joint lives, subtract the sum of the values of an annuity on the two joint lives AP and AQ, the difference will be the value required. See § 76.

Example. What is the value of an annuity on the life of a person aged 20, to be enjoyed by him after the decease of both his brother and sister, aged 30 and 40 respectively, interest at 4 per cent., and the probabilities of living as at *Northampton?*

The value of an annuity on the single life in reversion is, by Table X., equal to 16·003, and the value of an annuity on the three joint lives is, by Table XIV., equal to 8·986, the sum of which is 25·019; the value of an annuity on the two joint lives 20 and 30 is, by Table XIII., equal to 11·873; and the value of an annuity on the two joint lives 20 and 40 is equal to 10·924, the sum of which is 22·797; therefore 22·797 subtracted from 25·019 will leave 2·222 for the value required.

Had the two lives in possession been both 40, then 16·033 added to 8·601 (or the value of an annuity on three joint lives aged 20, 40, and 40, as found by Question V.) will make 24·634, from which we must subtract twice 10·924; the difference, or 2·786, will be the value in this case required.

QUESTION XV.

§ 396. To find the value of the *Reversion* of an annuity on the *longest* of two lives A and B, after any *single* life P.

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[1] Dr. Price has given a table of the value of reversionary annuities for the life of a wife after the death of her husband; both in *single* and *annual* payments during their joint lives: deduced from the *Sweden* observations and at 4 per cent. interest, according to the several ages therein mentioned. See his *Obs. on Rev. Pay.*, vol. ii. p. 431. The utility and convenience of the present rule, in enabling us to determine the propriety and efficacy of those schemes which are instituted for the *benefit of Widows*, will be shown in the following Chapter.

SOLUTION.

From the sum of the values of an annuity on each single life A and B in reversion and on the three joint lives, subtract the sum of the values of an annuity on each pair of joint lives AB, AP, BP; the difference will be the value required. See § 76.

Example. What is the value of an annuity on the longest of two lives aged 20 and 30, to be enjoyed after the extinction of a single life aged 40, interest at 4 per cent., and the probabilities of living as at *Northampton?*

By proceeding as in the last question, it will be found that the sum of the values of an annuity on the two single lives 20 and 30 is 30·814, that the value of an annuity on the three joint lives is 8·986, and that the sum of the values of an annuity on each pair of joint lives is 33·287; consequently 6·513 is the value required.

Had the two lives in reversion been both 20 years of age, then the sum of the values of an annuity on their single lives would be 32·066; the value of an annuity on the three joint lives would, by Question V., be 9·406; and the sum of the values of an annuity on each pair of joint lives would be 34·383; consequently 7·089 would be the value required.

On the Renewal of Leases for Lives.

§ 397. The three preceding questions will be found of great practical use in the *Renewal of Leases*[1] held on two or three lives, as they serve to show the value of the *Fine* that ought to be paid for *putting in* a new life in lieu of one that has *dropt* or become extinct. For, the value[2] of such fine is equal to the difference between the value of an annuity on the longest of all the lives (including the life or lives to be added) and the value of an annuity on the longest of the lives in possession, which rule will be found to agree with the solutions above given, according to the several cases there mentioned.[3]

Example 1. The value of the fine which ought to be given for *putting in* a new life, aged 20, to a lease held by Two lives, after One has dropped, is (supposing the existing life to be aged 40) equal to 4·079, or rather more than four years' purchase of the net improved rent of the estate,[1] as already found by Question XIII. Consequently, if the net improved rent of the estate had been £100 per annum, we should have £407, 18s. for the gross *sum* that ought to be paid down for the renewal required.

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[1] See more on the subject of the Renewal of Leases for lives, *and afterwards for a term certain,* in the observations at the end of Question XXIV.

[2] I call the *value* of a Fine, the *Number of years' purchase* that it is worth, agreeably to the principles laid down in deducing the value of annuities; see the remark in the note in page 214. This value, being multiplied by the net improved rent of the estate, will show the total sum of money that ought to be given for the renewal.

[3] This subject is more fully discussed in my *Tables for the Purchasing and Renewing of Leases,* 2d edit. 1807.

Example 2. The value of the fine which ought to be given for *putting in* a new life, aged 20, to a lease held by Three lives, after One has dropped, is (supposing the existing lives to be aged 30 and 40) equal to 2·222, or nearly 2¼ years' purchase of the net improved rent of the estate, as already found by Question XIV.

Example 3. The value of the fine which ought to be given for *putting in* Two new lives, both aged 20, to a lease held by Three lives, after Two have dropped, is (supposing the existing life to be aged 40) equal to 7·089, or rather more than seven years' purchase of the net improved rent of the estate, as already found by Question XV.

The same principles will also lead us to the true values that ought to be given for *Exchanging* any one or more lives (in a lease) for a life or lives of any other age. For, such value will in all cases be equal to the present value of the tenant's interest in the lease *before* the exchange, subtracted from his interest in the lease *after* the new lives are added.

QUESTION XVI.

§ 398. To find the value of the *Reversion* of an annuity on a *single* life, after the extinction of two *joint* lives.

SOLUTION.

From the value of an annuity on the single life in reversion subtract the value of an annuity on the three joint lives: the difference will be the value required. See § 76.

Example. What is the value of an annuity on the life of a person aged 20 to be enjoyed by him after the decease of either of his brothers, one aged 30 and the other 40; interest at 4 per cent., and the probabilities of living as at *Northampton*?

The value of an annuity on the life of a person aged 20 is, by Table X., equal to 16·033; and the value of an annuity on the three joint lives is, by Table XIV., equal to 8·986: therefore 7·047 is the value required.

Had the two lives in possession been both 40 years, then 7·432 would have been the value required.

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[1] That is, the net *surplus* rent, after deducting the reserved rent (if any), and all taxes and other annual charges.

QUESTION XVII.

§ 399. To find the value of the *Reversion* of an annuity on two *joint* lives after the extinction of a *single* life.

SOLUTION.

From the value of an annuity on the two joint lives in reversion, subtract the value of an annuity on the three joint lives: the difference will be the value required. See § 76.

Example. What is the value of the reversion of an annuity on two joint lives aged 20 and 30 after the extinction of a life aged 40; interest at 4 per cent., and the probabilities of life as at *Northampton?*

The value of an annuity on the two joint lives aged 20 and 30 is, by Table XIII., equal to 11·873, and the value of an annuity on the three joint lives is, by Table XIV., equal to 8·986; therefore 2·887 is the value required.

Had both the lives in reversion been 20 years of age, then 3·129 would be the answer required.

QUESTION XVIII.

§ 400. To find the value of any *Deferred reversionary* life annuity.[1]

SOLUTION.

Substitute the values of *deferred* annuities on each single and joint lives, instead of the annuities for the *whole continuance* of those lives; and proceed as in the solutions to the last five questions, according to the case. See § 77.

Example 1. What is the present value of a reversionary annuity on the life of a person aged 20, to commence at the end of thirty years, provided another person, now 40, be then dead; or if this should not happen, then at the end of any year in which the former shall happen to survive the latter; interest at 4½ per cent., and the probabilities of living as observed by *M. De Parcieux?*

The value of an annuity on the life in reversion, deferred for thirty years, is, by Question VI., equal to 2·272; and the value of an annuity on the two joint lives, deferred for thirty years, is, by the same question, equal to ·496. Therefore the difference between these two values, or 1·776, will, by Question XIII., be the answer required.

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[1] This question is of considerable use in enabling us to determine the validity of certain Schemes which have been proposed for providing Annuities for the *benefit of Widows;* as will be more fully explained in the following Chapter. See also the note in p. 230.

Example 2. A woman aged 40 will at the end of ten years enter upon an annuity for her life, provided her husband, now aged 46, be then dead; or if this should not be the case, then at the end of any year in which he may die: what is the present value of the reversion, interest at 4 per cent., and the probabilities of living as observed in *Sweden?*

The value of an annuity on the life of a female aged 40, deferred for ten years, is, by Question VI., equal to 6·926; and the value of an annuity on their joint lives is, by the same Question, equal to 3·511. Therefore the difference between these two values, or 3·415, will, by Question XIII., be the answer required.

If we wish to determine the value of these deferred reversionary annuities in *annual* payments during the continuance of their joint lives, we have only to divide the *single* payment, above found, by unity added to the value of an annuity on the joint lives, as already explained in the Scholium to Question XIII.

SCHOLIUM.

§ 401. If the deferred annuity mentioned in this question depends on the *joint* continuance of all the lives to the end of the given term, the solution will be materially different (as I have already observed respecting deferred annuities depending on the longest of any lives, in the Scholium in page 229); and care must be taken not to confound the two cases together. When the reversion depends on the *joint* continuance of all the lives to the end of the given term, its value will be equal to the value of the reversion on the same number of lives, each older by the given term than the given lives, multiplied by the expectation that the *joint* lives shall receive £1 at the end of that term. See § 78.

Example 1. What is the present value of a reversionary annuity on a life aged 20 for what may happen to remain of it beyond another life aged 40, after thirty years, provided *both* lives continue from the present time to the end of the term; interest at 4½ per cent., and the probabilities of living as given by *M. De Parcieux?*

The value of an annuity on a life aged 50 after another life aged 70 is, by the rule in Question XIII., equal to 6·904 (or the difference between 11·921 and 5·517); and the expectation that the joint lives, 20 and 40, shall receive £1 at the end of thirty years is, by Question II., equal to ·0899, the product of these two quantities therefore, or ·576, will be the value required.

Example 2. What is the present value of an annuity on the life of a woman aged 40, for what may happen to remain of it beyond the life of her husband, now aged 46, after ten years, provided they *both* continue in being so long; interest at 4 per cent., and the probabilities of life as observed in *Sweden?*

The value of an annuity on the life of a woman aged 50, after the decease of her husband aged 56, is, by the rule in Question XIII., equal to 4·175; and the expectation that the joint lives 40 and 46 shall receive £1 at the end of ten years is, by Question II., equal to ·4459; consequently, these two quantities multiplied together will produce 1·862 for the value in this case required.

If we wish to determine the value of these deferred reversionary annuities in *annual* payments during the continuance of their joint lives, we have only to divide the *single* payment above found, by unity added to the value of an annuity on the joint lives, as already explained in the Scholium to Question XIII.

QUESTION XIX.

§ 402. To find the value of any *Temporary reversionary* life annuity.

SOLUTION.

Substitute the values of *temporary* annuities on each single and joint lives, instead of the annuities for the *whole continuance* of those lives, and proceed as in the last question. See § 77.

Example 1. A lease of an estate is held for thirty years, to the rent of which a person now aged 20 will be entitled on the decease of his brother aged 40: what is the value of his interest therein, taking the probabilities of life as observed by *M. De Parcieux*, and interest at 4½ per cent.?

The value of an annuity for thirty years on a life aged 20 is, by Question VII., equal to 14·352, and the value of a similar annuity on the two joint lives is, by the same question, equal to 12·049; consequently, the difference of the two values, or 2·303, will, by Question XIII., be the answer required.

Example 2. In a lease of an estate (originally granted for twenty-one years) ten years are unexpired, to the rent of which a woman aged 40 will, on the decease of her husband aged 46, become entitled: what is the value of her interest in the same, taking the probabilities of life as observed in *Sweden*, and the rate of interest at 4 per cent.?

The value of an annuity on the life of a woman aged 40 for ten years is, by Question VII., equal to 7·475, and the value of a similar annuity on the two joint lives is, by the same Question, equal to 6·775; consequently ·700 is the value required.

If we wish to determine the value of these temporary reversionary annuities in *annual* payments, we have only to divide the single payments above found, by unity added to the value of a temporary annuity on the joint lives *for one year less than the given term*, and the quotient will be the annual payment required. See § 368.

QUESTION XX.

§ 403. Two persons, A and B, purchase an annuity on the longest of their lives, which is to be equally divided between them whilst they are both living, but on the decease of either of them it is to belong wholly to the *Survivor:* to find their respective shares, or the proportion which each person ought to contribute towards the purchase.

SOLUTION.

From the value of an annuity on the life A or B subtract half the value of an annuity on the two joint lives: the remainder will be the share of A or B required. See § 85.

Example 1. Suppose the age of A to be 20, and that of B 40; the rate of interest $4\frac{1}{2}$ per cent., and the probabilities of life as observed by *M. De Parcieux.*

The value of an annuity on a life aged 20 is, by Table X., equal to 16·624; and the value of an annuity on a life aged 40 is equal to 14·254; also the value of an annuity on two joint lives aged 20 and 40 is, by Table XI., equal to 12·545, the half of which is 6·272. Consequently this latter value subtracted from 16·624 will give 10·352 for the share which A ought to contribute; and when subtracted from 14·254 it will give 7·982 for the share which B ought to contribute.

Example 2. Suppose two persons, a man aged 46 and a woman aged 40, to hold the lease of an estate on the longest of their lives, the rent of which is divided in the manner above stated: what sum ought to be given to each of them for surrendering their right in the same; interest being reckoned at 4 per cent., and the probabilities of life as observed in *Sweden?*

The value of the man's interest is equal to 7·154 (or to the difference between 12·297 and 5·143); and the value of the woman's interest is equal to 9·258 (or to the difference between 14·401 and 5·143). Therefore if the net rent of the estate were £50 per annum, the sum which ought to be given to the man will be 357·700, or £357, 14s., and the sum which ought to be given to the woman will be 462·900, or £462, 18s.

SCHOLIUM.

§ 404. If the annuity is for a term of years, *less* than that to which it is probable the given lives may extend, we must substitute the values of annuities for the given *term* instead of the values of annuities for the *whole continuance* of the lives; and proceed with these substituted values according to the directions given in the rule.

Thus, if in the first example the annuity had been for thirty years only, we must find the value of a *temporary* annuity for thirty years on a single life aged 20, a single life aged 40, and two joint lives aged 20 and 40: which values are, by Question VII., equal to 14·352, 13·470, and 12·049 respectively. Consequently, the half of the latter subtracted from 14·352 will leave 8·328 for the share of A; and when subtracted from 13·470, it will leave 7·446 for the share of B.

QUESTION XXI.

§ 405. Two persons are in possession of an annuity on the longest of their lives; which, on the decease of either of them, will belong to D and his heirs during the life of the *Survivor:* to find the value of his interest therein.

SOLUTION.

From the sum of the values of an annuity on each single life in possession, subtract twice the value of an annuity on their joint lives: the difference will be the value required. See § 137.

Example. Suppose the ages of the two lives in possession to be 20 and 40; interest 4½ per cent., and the probabilities of living as observed by *M. De Parcieux.*

The value of an annuity on each single life is, by Table X., equal to 16·624 and 14·254; and the value of an annuity on the two joint lives is, by Table XI., equal to 12·545. Consequently, 25·090 subtracted from 30·878 will leave 5·788 for the interest of D and his heirs in this annuity.

Had the two lives in possession been a man aged 46 and a woman aged 40, the value of the interest of D and his heirs in the annuity would (on the supposition that interest was at 4 per cent., and the probabilities of living as observed in *Sweden*) be equal to 6·126.

SCHOLIUM.

§ 406. If the annuity is for a term of years *less* than that to which it is possible that either of the given lives may extend, we must substitute the values of annuities for the given *term* instead of the values of annuities for the *whole continuance* of the lives; and proceed with these substituted values according to the directions given in the rule.

Thus, if in the example just given the annuity on the two lives aged 20 and 40 had been for thirty years only, then 24·098 (or twice the value of a temporary annuity for thirty years on the two joint lives, as found by Question VII.) subtracted from 27·822 (or the sum of the values of a temporary annuity for thirty years on each of the single lives, as found by the same question) will leave 3·724 for the value in this case required.

QUESTION XXII.

§ 407. To find the value of an annuity *certain* for a given term; and afterwards, for the *remainder* of any given life or lives.

SOLUTION.

To the present value of an annuity certain for the given term add the value of an annuity on the given life, or lives, *deferred* for that term: the sum of these two will be the value required. See § 51.

Example 1. What is the value of an annuity *certain* for thirty years, and then to continue during the life of a person now aged 20; reckoning interest at 4½ per cent., and the probabilities of living as observed by *M. De Parcieux?*

The value of an annuity certain for thirty years is, by Table IV., equal to 16·289, and the value of an annuity on a life aged 20, deferred for thirty years, is, by Question VI., equal to 2·272: consequently 18·561 will be the value required.

Example 2. What is the value of an annuity *certain* for ten years, and then to continue during the joint lives of a man aged 46 and his wife aged 40; reckoning interest at 4 per cent., and the probabilities of living as observed in *Sweden?*

The value of an annuity certain for ten years is, by Table IV., equal to 8·111, and the value of an annuity on the two joint lives, deferred for ten years, is, by Question VI., equal to 3·511: consequently 11·622 will be the value required.

QUESTION XXIII.

§ 408. Supposing a person to enjoy an annuity for his life; and, at his decease, to have the nomination of a successor: to find the present value of the annuity on the *Succeeding life.*

SOLUTION.

Multiply the value of an annuity on the life in possession by the rate of interest, and subtract the product from unity; multiply the remainder by the assumed[1] value of an annuity on the succeeding life: the product will be the present value required. See § 205.

Example. Suppose the age of the life in possession to be 65, and that at his decease he has the liberty of nominating another life to succeed him; which life we will suppose to be one of the best that can *then* be found, or one which may *then* be about 10 years old: what is the *present* value of such succeeding life, interest at 4½ per cent, and the probabilities of living as observed by *M. De Parcieux?*

The value of an annuity on a life aged 65 is, by Table X., equal to 7·780, which being multiplied by ·045, and subtracted from unity, will leave ·6499; and this quantity, multiplied by 17·515 (or the value of an annuity on the life to be hereafter nominated) will produce 11·383 for the *present* value of the same, as was required.

Had it been required to calculate the value of the succeeding life according to the *Northampton* tables of observations, at 4 per cent. interest, we ought to multiply 7·761 by ·04; which, subtracted from unity, would leave ·68956: and this latter quantity multiplied by 17·662[2] will give 12·179 for the value in this case required.

SCHOLIUM.

§ 409. The solution given to the present question will apply equally to the case of annuities on *joint* lives, or on the *longest* of any lives, with power to nominate, at the extinction of such lives, an equal number of similar lives to succeed thereto.

Example. Suppose an annuity is held on two *joint* lives aged 50 and 60; and that, on the extinction of either of them, two other joint lives (the best that can then be found, and which we will suppose to be 10 years old) are nominated to succeed them: what is the present value of the annuity on the succeeding joint lives, interest at 4 per cent., and the probabilities of living as observed at *Northampton?*

The value of an annuity on the two *joint* lives is, by Table XIII., equal to 6·989; which, being multiplied by ·04 and subtracted from unity, will leave ·72044, and this multiplied by 14·277 (or the value of an annuity on two joint lives both 10 years old) will produce 10·286 for the value required.

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[1] The life or lives nominated to succeed to the annuity, after the extinction of the life or lives in possession, are such as are *then* to be fixed on at pleasure: therefore the present value of an annuity on those lives will vary according to the ages at which they are supposed to be put in. See p. 104.

[2] It appears that, by the *Northampton* tables, a life of the age of 8 years is one of the best that can be put in; since the value of an annuity on such life is equal to 17·662: but in questions of this kind we may safely omit the *decimal,* and assume the life to be such that an annuity on it at the time of nomination would be worth 17 years' purchase. For, it seldom happens that the life which we should *choose* to nominate is exactly of the age which is assumed.

Had the annuity been held on the *longest* of the two lives, aged 50 and 60; with power, on the extinction of *both* those lives, to nominate two other lives (whose ages we will suppose to be each 10 years) who are to enjoy the annuity as long as either of them is in existence; the present value of the annuity on those succeeding lives might be calculated in a similar manner. For, the value of an annuity on the longest of two lives aged 50 and 60 is, by the solution in Question VIII., equal to 13·314; which being multiplied by ·04 and subtracted from unity, leaves ·46744; and this quantity multiplied by 20·769 (or the value of an annuity on the longest of two lives aged 10 years) will give 9·708 for the value required.

QUESTION XXIV.

§ 410. To find the present value of an annuity *certain* for a given term *after* the extinction of any given life or lives.

SOLUTION.

Multiply the value of an annuity on the given life or lives by the rate of interest, and subtract the product from unity; multiply the remainder by the present value of an annuity certain for the given term: the product will be the value required. See § 208.

Example 1. Suppose D and his heirs to be entitled to an annuity certain for twenty-one years, to commence at the death of a person aged 70; what is the present value of D's interest in that annuity, taking the probabilities of living as observed at *Northampton*, and the rate of interest at 4 per cent.?

The value of an annuity on the life of a person aged 70 is, by Table X., equal to 6·361, which being multiplied by ·04 and subtracted from unity, will leave ·71376; and this multiplied by 12·821 (or the value of an annuity certain for twenty-one years) will produce 9·151 for the present value of the same annuity to be entered on[1] at the extinction of the given life.

If this value be added to 6·361 (or the value of the annuity on the life in possession) the sum of them, or 15·512, will be equal to the value of an annuity on the given life, commencing immediately, and to continue, after the extinction of such life, for the term of twenty-one years longer.

Example 2. A lease of an estate is held upon two lives aged 60 and 70; and after the decease of both of them, then for twenty-one years longer: what is the value of such lease, reckoning interest at 4 per cent., and the probabilities of living as at *Northampton*?

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[1] This solution supposes that the first payment of the annuity is made at the end of the year in which the given life becomes extinct.

The value of an annuity on the longest of two lives aged 60 and 70, is, by the rule in Question VIII., equal to 10·500; which being multiplied by ·04, and subtracted from unity, will leave ·58000; this being multiplied by 12·821 (or the value of an annuity certain for twenty-one years) will give 7·436 for the present value of the same annuity to be enjoyed twenty-one years after the extinction of the longest of the two lives. And this value being added to 10·500 (or the value of an annuity on the longest of the two lives) will give 17·936 for the value of the lease required.

Example 3. A lease of an estate is held upon three lives, aged 50, 60, and 70, and after their decease, then for twenty-one years longer: what is the value of the same, reckoning interest at 4 per cent., and the probabilities of living as at *Northampton?*

The value of an annuity on the longest of three lives, aged 50, 60, and 70, is, by the rule in Question IX., equal to 13·688, which being multiplied by ·04 and then subtracted from unity, will leave ·45248; this being multiplied by 12·821 will give 5·557 for the present value of the annuity for twenty-one years after the longest of the three lives. And this value, being added to 13·688, will give 19·245 for the value of the lease required.

Had the three lives been 10, 60, and 70 years of age, then the value of an annuity on the longest of their lives would be equal to 18·376;[1] and the value of the annuity for twenty-one years after those lives would be equal to 1·041; consequently the value of the lease would in this case be equal to 19·417.

Or had the three lives been 10, 10, and 70 years of age, then the value of an annuity on the longest of their lives would be equal to 18·801;[2] and the value of the annuity for twenty-one years after those lives would be equal to 1·128; consequently the value of the lease would in this case be equal to 19·929.

On the Renewal of Leases for lives, and afterwards for a Term certain.

§ 411. The three examples given in the preceding Question will serve to show the method of determining the value of the *Fine* which ought to be given for *Renewing* any lives dropt in a lease originally held on three lives, *and for a term certain after the extinction of those lives.* For, the value of such fine will in all cases be equal to the present value of the tenant's interest in the lease *before* the renewal, subtracted from his interest in the lease *after* the new lives are added.[3]

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[1] The value of an annuity on these three *joint* lives is, by the rule in Question V. (and the correction alluded to in page 221) equal to 4·775.

[2] The value of an annuity on these three *joint* lives is, by the rule in Question V. (and the correction alluded to in page 221) equal to 4·932.

[3] See what has been already said on the subject of the Renewal of Leases for Lives in general, in § 397.

Thus, suppose that in a lease originally held on three lives and twenty-one years, *one* of the lives has dropt, and that the ages of the two remaining lives are 60 and 70, the value of the fine which ought to be paid for putting in another life aged 10 years is equal to the difference between 17·936 (or the value found by the second example) and 19·417 (or the value found by the second case in the third example). That is, the value of the fine will be equal to 1·481, or about 1½ years' purchase of the net improved rent of the estate.

Again, let us suppose that *two* of the lives have dropt, and that the age of the remaining life is 70; the value of the fine which ought to be paid for putting in two other lives both 10 years of age, is equal to the difference between 15·512 (or the value found by the first example) and 19·929 (or the value found by the third case in the third example); consequently the value of the fine will be equal to 4·417, or about 4⅖ years' purchase of the net improved rent of the estate.

These examples will also serve to show the sum that ought to be given for *Exchanging* any of the lives on which the lease may happen to be held; for, the same method of solution will apply to such cases. Thus, suppose that in a lease held on three lives and twenty-one years, the ages of the lives at present in the lease were 50, 60, and 70, and that the tenant is desirous of exchanging the life of 50 for another life aged 10 years old; the value of the fine which ought in such case to be paid will be equal to the difference between 19·245 (or the value of his *present* interest, as found by the first case in the third example) and 19·417 (or the value of his interest *after* the exchange, as found by the second case in the same example). That is, the value of the fine will be 0·172, or near ⅙ year's purchase of the net improved rent of the estate.

§ 412. Many of the estates belonging to the Corporation of *Liverpool* are held on the tenure alluded to in these examples, and till lately they were in the constant habit of renewing their leases on the following terms, viz., One year's purchase for adding one life dropt; Three years' purchase for adding *two* lives dropt; and Seven years' purchase for adding *three* lives dropt, when the twenty-one years remain unexpired. In all these cases no regard was paid to the *age* or state of health of the existing lives in the lease. This practice of demanding a uniform fine for renewing with *any* life, and without regard to the age or state of health of the lives remaining in the lease, betrayed a total want of knowledge on the subject, and was in most cases injurious to the interests of the Corporation.

But the most singular circumstance attending this subject was their custom of *exchanging*, for the sum of only *one guinea* each, lives not exceeding 50 years of age and in good health, for lives of *any other age*, and in estates of *any* yearly value! A practice which could hardly be supposed ever to have existed in so enlightened a place as *Liverpool*. The Corporation, at length suspecting that their mode of proceeding was incorrect in principle, referred the matter to a committee, who directed it to be laid before me for my opinion; and, agreeably to their request, I calculated a set of tables for their use, founded on the principles detailed in the preceding examples.

As it is probable that many other corporate bodies are still pursuing the same incorrect and absurd practice of leasing their estates, I have been more particular in these examples, in order that they may the more readily determine the values that *ought* to be given in such cases.

QUESTION XXV.

§ 413. To find the present value of what may happen to *remain* (after any given life or lives) of an annuity *certain* for a given term, provided such term be *less* than that to which it is possible the given lives may extend.[1]

SOLUTION.

From the value of an annuity certain for the given term, subtract the value of an annuity on the given life or lives *for the given term*, the difference will be the value required. See § 192.

Example. A lease of an estate is held for thirty years, to the rent of which a person aged 20 is entitled, provided he lives so long; but if not, then the remainder of the lease will descend to his heirs: what is the value of their interest in the same, taking the probabilities of living as observed by *M. De Parcieux*, and reckoning interest at the rate of 4½ per cent.?

The value of an annuity certain for thirty years is, by Table IV., equal to 16·289; and the value of a temporary annuity for thirty years on a life aged 20 is, by Question VII., equal to 14·352, therefore this latter quantity subtracted from the former will leave 1·937 for the value of the reversion required.

Had the life been 40 years of age, the value would have been equal to the difference between 16·289 and 13·470; that is, 2·819 would be the value of the reversion in this case required.

Had these two *joint* lives (aged 20 and 40) been entitled to the rent of the estate provided they lived so long, then 12·049 (or the value of a temporary annuity for thirty years on those two *joint* lives, as found by Question VII.) subtracted from 16·289, would give 4·240 for the value of the reversion in this case required.

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[1] The term to which it is possible that any given life or lives may extend is—for a *single* life, equal to the difference between the age of such life and the age of the oldest life in the table of observations; for *joint* lives, equal to the difference between the *oldest* of such lives and the age of the oldest life in the table of observations; and for the *longest* of any number of lives, equal to the difference between the *youngest* of such lives and the age of the oldest life in the table of observations.

Or, had the *longest* of these two lives been entitled to the rent of the estate, then 15·773 (or the value of a temporary annuity for 30 years on the *longest* of those lives, as found by Question XII.) subtracted from 16 289, would give ·516 for the value of the reversion in this case required.

QUESTION XXVI.

§ 414. To find the value of the *Assurance* of an *estate* (or annuity certain for any given term[1]) to be entered upon at the extinction[2] of any given lives.

SOLUTION.

Subtract the value of an annuity on the given lives[3] from the value of the perpetuity, or the terminable annuity, and the difference will be the value required. See § 189.

Example 1. What is the value of the reversion of an estate in fee after the death of a person now aged 20; interest being reckoned at 4½ per cent., and the probabilities of life as observed by *M. De Parcieux?*

The value of the perpetuity is, by Table IV., equal to 22·222, and the value of an annuity on the life of a person aged 20 is, by Table X., equal to 16·624; consequently the difference between these two values, or 5·598, will be the answer required. Therefore if the estate produced a rent of £4, 10s. per annum, its present value in a *single* payment would be 25·191, or £25, 3s. 10d.

This is the true present value of the assurance in a *single* payment; but in order to obtain the value of the same in *annual* payments, commencing immediately, we must divide the sum thus found by unity added to the value of an annuity on the given life (agreeably to the principles laid down in § 369), and the quotient will be the answer required.

Thus, in the present case, if we divide 25·191 by 17·624, the quotient will be 1·429, or £1, 8s. 7d.; and this is the sum that ought to be paid annually during the life of the person assured, in order to secure the perpetuity of £4, 10s. per annum on his death, the first of those annual payments being made *immediately*, and the remaining ones at the *beginning* of every subsequent year.

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[1] Provided such term be *not less* than that to which it is probable the given lives may extend. For, in *such* case, the solution is obtained by the preceding Question.

[2] That is, the first payment of the annuity is to be made at the end of the year in which such lives become extinct, and this is always understood in questions of this kind.

[3] Whether a *single* life, or *joint* lives, or the *longest* of any number of lives, for the solution will apply to each case.

Had the question referred to an *annuity* for eighty years[1] instead of a perpetuity, then 16·624 subtracted from 21·565 (or the present value of an annuity certain for that term by Table IV.) would leave 4·941 for the answer required. Therefore, if the annuity, as in the preceding case, were £4, 10s. per annum, its present value in a *single* payment would be 22·234; and this sum, divided by 17·624, would give 1·262 for the value of the same in *annual* payments.

Example 2. What is the value of a freehold estate to be entered upon at the death of *either* of two lives, a man aged 46 and a woman aged 40; reckoning interest at 4 per cent., and the probabilities of life as observed in *Sweden?*

The value of the perpetuity is, by Table IV., equal to 25, and the value of an annuity on the joint lives of these two persons is, by Table X., equal to 10·286; this latter value subtracted from the former will leave 14·714 for the answer required. Therefore if the estate produced £4 per annum, its present value would be 58·856, or £58, 17s. 1d., in a *single* payment; and this sum, divided by 11·286, would give 5·215, or £5, 4s. 4d., for the value of the same in *annual* payments.

If the estate were not to be entered upon till the extinction of *both* the lives, then 16·412 (or the value of an annuity on the *longest* of the two lives, as found by Question VIII.) subtracted from 25, will leave 8·588 for the number of years' purchase required; and which being multiplied by 4, as in the preceding case, will give 34·352 for the value of the same estate in a *single* payment; and this sum, divided by 17·412, will give 1·973 for the value of the same in *annual* payments.

Had it been a leasehold estate of £4 per annum for sixty years, instead of a freehold, the value would, in the former case, have come out equal to 49·348 in a *single* payment; or 4·373 in *annual* payments. And in the latter case, to 24·844 in a *single* payment; or 1·427 in *annual* payments.

QUESTION XXVII.

§ 415. To find the value of an *Assurance* of a given *sum*, which is to be received on the extinction[2] of any given lives.

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[1] It must be particularly observed that, when we have to determine by this rule the value of the reversion of any terminable annuity after any given lives, the number of years during which such annuity is to continue must not be *less* than that to which it is probable the given lives may extend. See the note in p. 245.

[2] That is, at the end of the year in which such lives become extinct: and this is always understood in questions of this kind. The usual practice of the Assurance Offices, however, is to pay the sum at the end of six months from the time of the decease.

SOLUTION.

Multiply the value of an annuity on the given lives[1] by the rate of interest, and subtract the product from unity; divide the remainder by the amount of £1 in one year; and the quotient, multiplied by the given sum, will be the value required. See § 180.

Example 1. What is the present value of an assurance of £100 on the life of a person aged 20, interest at 4½ per cent., and the probabilities of living as observed by *M. De Parcieux*?

The value of an annuity on such life is, by Table X., equal to 16·624, which being multiplied by ·045 (or the rate of interest) will produce ·74808; the difference between this value and unity is ·25192, which being divided by 1·045 (or the amount of £1 in one year) will give ·24107 for the present value of £1 to be received on the extinction of the given life:[2] and this value, being multiplied by 100, will give 24·107, or £24, 2s. 2d., for the answer required, in a *single* payment.

But, in order to obtain the value of the same in *annual* payments, commencing immediately, we must divide the sum, thus found, by unity added to the value of an annuity on the given life; agreeably to the principles laid down in § 369: whence 24·107 divided by 17·624 will give 1·368, or £1, 7s. 4d., for the sum which ought to be paid annually during the life of the person assured, in order to secure the sum of £100 on his death. The first of those annual payments being made *immediately*, and the remaining ones at the *beginning* of every subsequent year.[3]

Had the life been 40 years of age, its value in a *single* payment would have been equal to 34·313; which being divided by 15·254 (or unity added to the value of an annuity on the given life) will give 2·249, or £2, 5s. 0d., for the value of the same in *annual* payments.

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[1] Whether a *single* life, or *joint* lives, or the *longest* of any number of lives; for the solution (as in the preceding question) will apply to each case.

[2] From the present value of *one pound* to be received on the extinction of any given life or lives, we may readily determine the sum, which ought to be paid, on the extinction of such lives, for *any given sum now advanced:* viz., by dividing this latter sum by the present value of £1 as above found. Thus, if a person, aged 20, borrows £4000, and gives security to pay the value of the same at his death, the sum which ought *then* to be paid (supposing the interest, &c., the same as mentioned in the text) is found by dividing 4000 by ·24107; which gives 16592·691, or £16,592, 13s. 10d., for the answer required. This method is universal; and will apply to all the subsequent questions in the present Chapter.

[3] The rates of Assurances for Lives, at *all* the different offices established in London, are calculated from the *Northampton* Table of Observations, and at 3 per cent. interest. By comparing these rates, both for *Single* and *Joint* lives, with the true and proper values, the public may form a tolerably accurate idea of the immense profit which is made by the several Assurance Companies above alluded to. See also the Scholium to Question XXIX., and the Scholium to Question XXX.

Example 2. What is the present value of £100 to be received on the death of a man aged 46, interest being reckoned at 4 per cent., and the probabilities of life as observed in *Sweden?*

The value of an annuity on the life of a man aged 46 is, by Table X., equal to 12·297; which being multiplied by ·04, and the product subtracted from unity, leaves ·50812; this quantity, divided by 1·04, gives ·48858; which, being multiplied by 100, produces 48·858, or £48, 17s. 2d., for the answer required, in a *single* payment. And if this latter sum be divided by 13·297, it will give 3·674, or £3, 13s. 6d., for the value of the same in *annual* payments. Had the age of the man been 56, the value of the assurance in a *single* payment would have been equal to 58·781; which, divided by 10·717, would give 5·485 for the value of the same in *annual* payments.

But if the sum had depended on the death of a woman aged 40, its value in a *single* payment would have been equal to 40·765, or £40, 15s. 4d. And this sum, divided by 15·401, would give 2·647, or £2, 12s. 11d., for the value of the same in *annual* payments. And had the age of the woman been 50, the value of the assurance in a *single* payment would have been equal to 49·812, which divided by 13·049 would give 3·817 for the value of the same in *annual* payments.

Example 3. What sum ought to be given for the assurance of £100 on two *joint* lives aged 20 and 40, interest at 4½ per cent., and the probabilities of living as observed by *M. De Parcieux?*

The value of an annuity on the two joint lives is, by Table XI., equal to 12·545, which being multiplied by ·045 (or the rate of interest) will produce ·56452; this quantity, subtracted from unity, leaves ·43547, which being divided by 1·045 (or the amount of £1 in the year) will give ·41672; and this multiplied by 100 will produce 41·672, or £41, 13s. 5d., for the answer required, in a *single* payment. If this latter quantity be divided by 13·545, it will give 3·077, or £3, 1s. 6d., for the value of the same in *annual* payments.

Had the two lives been 50 and 70, the value in a *single* payment would have been equal to 71·936; which being divided by 6·517 would give 11·038 for the value of the same in *annual* payments.

Had the assurance been made on the joint lives of a man aged 46, and his wife aged 40, the value of the same (reckoning interest at 4 per cent., and the probabilities of life as observed in *Sweden*) would have been equal to 56·592 in a *single* payment; and which being divided by 11·286, would give 5·014 for the value in *annual* payments.

Or, had these two lives been respectively 56 and 50, the value would in such case have come out equal to 65·869 in a *single* payment; and which being divided by 8·874 would give 7·423 for the value in *annual* payments.

Example 4. What is the value of an assurance of £100 on the *longest* of two lives aged 20 and 40, interest 4½ per cent., and the probabilities of life as observed by *M. De Parcieux?*

The value of an annuity on the longest of two lives aged 20 and 40 is, by Question VIII., equal to 18·333; which being multiplied by ·045 and subtracted from unity, leaves ·17502; this quantity divided by 1·045 will give ·16748, and which being multiplied by 100 will produce 16·748, or £16, 15s., for the answer required, in a *single* payment. If this latter quantity be divided by 19·333 (or unity added to the value of an annuity on the *longest* of the two lives) it will give ·866, or 17s. 4d., for the value of the same in *annual* payments, to be made at the beginning of each year during the continuance of either of the given lives.

Had the two lives been 50 and 70 years of age, the value in a *single* payment would have been equal to 41·328; which being divided by 13·625 would give 3·033 for the value of the same in *annual* payments.

Or, had the assurance been made on the longest of the two lives, of a man aged 46 and his wife aged 40, the value of the same (reckoning interest at 4 per cent., and the probabilities of life as observed in *Sweden*) would have been equal to 33·031 in a *single* payment; and which being divided by 17·412 would give 1·897 for the value of the same in *annual* payments.

And had these two lives been respectively 56 and 50, the value would in such case have come out equal to 42·723 in a *single* payment; which being divided by 14·892 will give 2·869 for the value of the same in *annual* payments.

Example 5. What is the present value of a legacy[1] of £100, to be received on the extinction of any *one* of three lives aged 20, 30, and 40; reckoning interest at 4 per cent., and the probabilities of living as at *Northampton?*

The value of an annuity on the three joint lives is, by Table XIV., equal to 8·986, which being multiplied by ·04, and the product subtracted from unity, leaves ·64056; this quantity divided by 1·04 will give ·61592, and which being multiplied by 100 will produce 61·592, or £61, 11s. 10d., for the present value of the legacy required.

In like manner we might determine the value of the legacy payable on the extinction of any *two* of the three lives above mentioned. For the value of an annuity on any two out of those three lives is, by Question X., equal to 15·315: consequently, by proceeding as in the last case, we shall find that 37·250, or £37, 5s., will be the present value of the legacy in this case required.

So also we might find the value of the legacy payable on the extinction of all the *three* lives. For, the value of an annuity on the longest of the lives is, by Question IX., equal to 19·710; and by proceeding in a similar manner it will be found that 20·346, or £20, 6s. 11d., is the present value of the legacy in this case required.

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[1] I consider a legacy as not due till the *end* of the year in which the testator dies, for it is seldom paid *immediately*.

SCHOLIUM.

§ 416. It may be here necessary to advert to the remark which I have made, in the Scholium in page 103, respecting the relative values of a reversionary *sum*, and a corresponding reversionary *estate*, and which may be verified by a comparison of the values in any two similar cases found by the rules in the two preceding Questions. Thus, by the first example in Question XXVII., it appears that the value of £100 payable on the decease of a person aged 20, interest at 4½ per cent., is equal to 24·107 pounds; and by the first example in Question XXVI., it appears that the value of a corresponding estate (or perpetuity) of £4, 10s. per annum is equal to 25·191 pounds. But the latter is to the former value in the proportion of 1·045 to 1, and *vice versa*, the former is to the latter value as 1 is to 1·045.

QUESTION XXVIII.

§ 417. To find the value of a *Deferred assurance* of any given sum on any given lives.

SOLUTION FIRST.

For Single and Joint Lives.

Find the value of the assurance of the given sum on the same number of single or joint lives as the given lives, but each older than such lives by the term given; multiply this value by the expectation that the given single or joint lives will receive £1 at the end of that term; the product multiplied by the given sum will be the value required. See note 2 in page 96.

Example 1. What is the present value of £100 to be received on the death of a man aged 46, provided that should happen *after* ten years, interest being reckoned at 4 per cent., and the probabilities of life as observed in *Sweden?*

The value of an assurance of £100 on a man aged 56 is, by Question XXVII., equal to 58·781; and the expectation that a man aged 46 will receive £1 at the end of ten years is, by Question II., equal to ·5241; consequently these two quantities being multiplied together will produce 30·807, or £30, 16s. 2d. for the answer required.

Had the assurance been on a woman aged 40, its present value would have been equal to 49·812 multiplied by ·5748; that is, equal to 28·632.

Example 2. What is the present value of £100 to be received on the extinction of *either* of two lives aged 20 and 40, provided that should happen *after* thirty years; interest at 4½ per cent., and the probabilities of life as observed by *M. De Parcieux?*

The value of the assurance on two joint lives aged 50 and 70 is, by Question XXVII., equal to 71·936; and the expectation of two joint lives, aged 20 and 40, receiving £1 at the end of thirty years is, by Question II., equal to ·0899; therefore the product of these two quantities will give 6·467, or £6, 9s. 4d., for the answer required.

Had the assurance been made on the joint lives of a man aged 46 and his wife aged 40, provided they became extinct *after* ten years, its present value (taking interest at 4 per cent., and the probabilities of life as observed in *Sweden*) would have been equal to 65·869 multiplied by ·4459; that is, equal to 29·371.

SOLUTION SECOND.

For the longest of any Lives.

§ 418. Multiply the value of an annuity on the longest of the given lives, *deferred* for the given term, by the rate of interest; subtract the product from the expectation of the longest of the given lives receiving £1 at the end of the same period, and divide the difference by the amount of £1 in a year: the quotient multiplied by the given sum will be the answer required. See § 182.

Example 3. What is the value of £100 to be received on the extinction of the longest of two lives, aged 20 and 40, provided that shall happen *after* thirty years; interest at 4½ per cent., and the probabilities of living as observed by *M. De Parcieux?*

The value of a deferred annuity for thirty years on the longest of the two lives is, by Question XI., equal to 2·560, which being multiplied by ·045 produces ·1152; and this subtracted from ·2266 (or the expectation that one or other of these two lives will receive £1 at the end of thirty years, as found by Question II.) will leave ·1114, which being divided by 1·045 will give ·10660; and this last value multiplied by 100 will produce 10·660, or £10, 13s. 2d., for the answer required.

Had this sum depended on the extinction of the longest of two lives, a man aged 46 and a woman aged 40, provided that event happened *after* ten years, the value of the same (reckoning interest at 4 per cent., and the probabilities of life as observed in *Sweden*) may be found in a similar manner. For the value of a deferred annuity for ten years on the longest of these lives is, by Question XI., equal to 8·508, which being multiplied by ·04 produces ·3403; and this subtracted from ·6530 (or the expectation that one at least of the two lives will receive £1 at the end of ten years, as found by Question II.) will leave ·3127, which being divided by ·104 will give ·30067; and this last value, multiplied by 100, will produce 30·067, or £30, 1s. 4d. for the answer required.

QUESTION XXIX.

§ 419. To find the value of a *Temporary assurance* of a given sum on any given lives.

SOLUTION.

From the value of the assurance of the given sum on the *whole continuance* of the given lives, subtract the value of a *deferred* assurance of the same sum for the given term, the difference will be the value required. See § 184.

Example 1. What is the present value of £100 to be received on the death of a man aged 46, provided that shall happen *within* ten years; interest being reckoned at 4 per cent., and the probabilities of life as observed in *Sweden*?

The value of an assurance of £100 on the whole continuance of this life is, by Question XXVII., equal to 48·858; and the value of a similar assurance, deferred for ten years, is, by Question XXVIII., equal to 30·807, which being subtracted from the former value will leave 18·051, or £18, 1s. 0d., for the answer required.

This is the value in a *single* payment; but if we wish to find the corresponding value in *annual* payments, we must divide this sum by 7·680 (or unity added to the value of an annuity on the given life for one year less than the given term, as found by the rule given in note 1, page 224), which will give 2·350, or £2, 7s. 0d., for the value of the same assurance in *annual* payments.

Had the assurance been made on a woman aged 40, its present value would have been equal to the difference between 40·765 and 28·632; that is, 12·133, or £12, 2s. 8d., would have been the answer required in a *single* payment. And this value divided by 7·900 (or unity added to the value of an annuity on the life for *nine* years, as found by the rule above mentioned) will give 1·536, or £1, 10s. 9d., for the value of the same in *annual* payments.

Example 2. What is the value of an assurance of £100 on the *joint* lives of two persons, aged 20 and 40, for *thirty* years; interest at 4½ per cent., and the probabilities of living as observed by *M. De Parcieux*?

The value of the assurance on the whole continuance of the lives is, by

Question XXVIII., equal to 41·672; and the value of a similar assurance deferred for thirty years is, by Question XXVIII., equal to 6·467; therefore this latter value, subtracted from the former, will leave 35·205, or £35, 4s. 1d., for the answer required, in a *single* payment. And this sum divided by 12·959 (or unity added to the value of an annuity on the two joint lives for twenty-nine years) will give 2·717, or £2, 14s. 4d., for the value of the same assurance in *annual* payments.

Had the assurance been made for ten years on the joint lives of a man aged 46 and his wife aged 40, the value of the same (reckoning interest at 4 per cent., and the probabilities of living as observed in *Sweden*) would have been equal to the difference between 56·592 and 29 371; that is, 27·221 would have been the value in a *single* payment. And this sum divided by 7·239 (or unity added to the value of an annuity on the two joint lives for nine years) will give 3·714 for the value of the *annual* payments.

Example 3. What sum ought to be given for the assurance of £100 for thirty years on the *longest* of two lives, aged 20 and 40; interest at 4½ per cent., and the probabilities of life as observed by *M. De Parcieux?*

The value of the assurance for the whole continuance of the given lives is, by Question XXVII., equal to 16·748; and the value of a similar assurance, deferred for thirty years, is, by Question XXVIII., equal to 10·660; this latter value therefore subtracted from the former will leave 6·088, or £6, 1s. 9d., for the answer required in a *single* payment. And this sum divided by 16·546 (or unity added to the value of an annuity on the longest of the two lives for twenty-nine years) will give ·368, or 7s. 4d., for the value of the same in *annual* payments.

Had it been required to determine the value of the assurance, in annual payments during the *joint* continuance of the given lives, then 6·086 divided by 12·959 (or unity added to the value of an annuity on the two joint lives for twenty-nine years) would give ·470 for the answer required.

In like manner may be determined the value of an assurance of £100 for ten years on the longest of two lives, viz., a man aged 46 and a woman aged 40; reckoning interest at 4 per cent., and the probabilities of life as observed in *Sweden.*

For, the value of such assurance will be equal to the difference between 33·031 and 30·067; that is, 2·964 will be the value in a *single* payment; and this sum divided by 8·251 (or unity added to the value of an annuity on the *longest* of the two lives for nine years) will give ·359 for the value of the same in *annual* payments during the *longest* of the given lives; or being divided by 7·329 (equal to unity added to the value of an annuity on the two *joint* lives for nine years) will give ·404 for the value of the same in *annual* payments during the existence of the *joint* lives.

SCHOLIUM.

§ 420. When we have to determine the value of a temporary assurance for a very short term, such as one, two, three, &c., years, it will be the easiest method to calculate the value of each year's expectation from the tables of mortality. For the probability that a person of any given age will die in any particular year is a fraction whose denominator is the number of persons living at that age, and whose numerator is the number of persons that die within the given year; and which fraction, being multiplied by the present value of the given sum due at the end of the given year, will give the expectation of receiving such sum at the end of that year, provided the given life becomes extinct in that year; and the sum of these annual expectations for the first, second, third, &c., years will be the value of the assurance for those periods respectively.

Example 1. What is the value of an assurance of £100, for *one* year, on the life of a woman aged 40; or, in other words, what is the present value of £100 to be received at the end of the year, provided such life be then extinct; interest being reckoned at 4 per cent., and the probabilities of life as observed in *Sweden*?

The probability that a woman of 40 will die within the *first* year is, by Table VIII., equal to $\frac{65}{4733}$; and the present value of £100 to be received at the end of a year is, by Table III., equal to 96·154; these two quantities, multiplied together, will produce 1·321, or £1, 6s. 5d., for the value required.

In like manner it may be found that the probability of a woman, aged 40, dying within the *second* year is, by the same table, equal to $\frac{75}{4733}$; and that the present value of £100 to be received at the end of two years is equal to 92·456; which quantities being multiplied together will produce 1·465 for the present value of £100 to be received at the end of the second year, provided the given life becomes extinct in that year. And this value, added to the one above found, will give 2·786, or £2, 15s. 9d., for the value of the assurance for *two* years.

By a similar method of proceeding it will be found that $\frac{76}{4733}$ multiplied by 88·900 will give 1·428 for the present value of £100 to be received at the end of the *third* year on a similar contingency; and which value, being added to the sum of the two former ones, will give 4·214 for the value of the assurance for *three* years. And so on for the subsequent years.

Had it been required to determine the value of a similar assurance on the life of a man, aged 46, the expectations for the first, second, and third years would have been equal to $\frac{80}{3991}$ multiplied by 96·154, 92·456, and 88·900 respectively;[1] whence, those expectations would have come out equal to 1·927, 1·853, and 1·782; and the value of the assurance, for one, two, and three years, would have been 1·927, 3·780, and 5·562 respectively.

The same observations will apply to assurances for one year on any *joint* lives. For, the probability that any two joint lives will fail within the first year is the difference between unity and the product of the probabilities that they shall both live to the end of the year; and which difference, being multiplied by the present value of the given sum due at the end of that year, will give the expectation of receiving such sum at that period, provided either of the given lives be then extinct.

Example 2. What is the present value of an assurance of £100 for *one* year on the joint lives of a man aged 46 and his wife aged 40, interest at 4 per cent., and the probabilities of life as observed in *Sweden?*

The probability that a man aged 46 will live to the end of the year is equal to $\frac{3911}{3991}$; and the probability that a woman aged 40 will live to the end of the same period is equal to $\frac{4668}{4783}$; these two fractions, therefore, being multiplied together and their product subtracted from unity, will leave ·0335; which being multiplied by 96·154 (or the present value of £1 due at the end of the year) will produce 3·221, or £3, 4s. 5d., for the value required.

§ 421. These examples will show the method of proceeding in all similar cases; and for the information of the reader, I shall here subjoin a table of the sums demanded by the different Assurance Companies for the assurance of £100 for *one year* on a single life at the several ages therein mentioned; to which I shall add the fair value that *ought* to be given for the same, according to the probabilities of life as observed by *M. De Parcieux*,[2] and reckoning interest at 4 per cent.

Ages.	Northampton, 3 per cent.	De Parcieux, 4 per cent.
10	·890	·929
20	1·362	·900
30	1·661	1·037
40	2·030	1·049
50	2·753	1·431
60	3·906	2·983
70	6·184	5·289

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[1] Because $\frac{80}{3991}$ is not only the probability that such life will die in the *first* year; but also the probability that it will die in the *second* year; and also in the *third* year; as may be seen by Table VIII.

[2] The probabilities here alluded to are, in this particular case, taken from the Table of Observations given by Dr. Price in his *Obs. on Rev. Pay.*, vol. ii. p. 456; because the *decrements* of life are there more correctly given; and being on a more enlarged scale, are therefore more applicable to the present examples.

From which it appears that the several assurance companies require, in most cases, half as much again as ought to be given; and in some cases nearly *double* the sum that should be given for the assurance. And though some compensation ought to be allowed for the expenses incurred in carrying on the business of the office, as well as a proper remuneration for the services of those who conduct it; yet it is evident that these sums are greater than ought reasonably to be taken; particularly when it is considered that those who insure at *any*[1] of the offices, for a *term* of years only, have not much prospect of deriving any advantage from the profits of the concern.

QUESTION XXX.

§ 422. To find the value of an *Assurance* of a given sum to be received on the decease of A, provided he dies *before* another given life B.[2]

SOLUTION.

Let O represent a life one year *older* than A; and Y a life one year *younger* than A. Add unity to the value of an annuity on the two joint lives OB, and multiply the sum by the number of persons living at the age of O; then divide this product by the amount of £1 in a year, and reserve the quotient. Multiply the value of an annuity on the two joint lives YB by the number of persons living at the age of Y; and, having subtracted the product from the reserved quotient, divide the remainder by the number of persons living at the age of A. Subtract this last quotient from the present value of £1 payable on the extinction of the two joint lives AB; and the remainder, multiplied by half the given sum, will be the value required. See § 231.

Example 1. What is the present value of £100 payable on the death of A, aged 20, provided B, aged 40, be then living; interest at $4\frac{1}{2}$ per cent., and the probabilities of living as observed by *M. De Parcieux*?[3]

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[1] For such persons do not (even at the Equitable Society) participate in the profits, unless a bonus happens to be declared during the term for which they are assured; which, in most ordinary cases (if it occurs at all) is but a *partial* advantage.

[2] The present question will be found of considerable utility in enabling us to determine the propriety and advantage of those schemes which are formed with a view of providing *sums* of money to be paid to *Widows* on the decease of their husbands.

[3] When the two lives are of the *same* age, the present value required is, in all cases, equal to the present value of half the sum payable on the extinction of the two joint lives AB. Thus, if the two lives, in the first example, were both aged 20, the present value required would be equal to 17·695; and if they had both been 40 years of age, the required value would have been equal to 22·634.

The value of an annuity on the two joint lives OB, aged 21 and 40, is, by the rule in page 216, equal to 12·520; which being added to unity, and then multiplied by 806 (or the number of persons living against the age of 21, in Table VII.), will produce 10897·120; and this being divided by 1·045 will give 10427·866 for the reserved quotient. The value of an annuity on the two joint lives YB, aged 19 and 40, is, by the rule in page 216, equal to 12·575; which being multiplied by 821 (or the number of persons living at the age of 19, in Table VII.) will produce 10324·075; this being subtracted from 10427·866 (the reserved quotient) and the remainder divided by 814 (or the number of persons living at the age of 20) will give ·1275. Now, the value of £1 payable on the extinction of the joint lives AB., aged 20 and 40, is, by Question XXVII., equal to ·4167; therefore ·1275 being subtracted from this value will leave ·2892; which being multiplied by 50 will give 14·46, or £14, 9s. 2d., for the present value required.

Having thus found the present value of the given sum payable on the decease of A, provided B be then alive, we may easily determine the present value of the same sum payable on the decease of B, provided A be then alive. For we have only in such case to deduct the value, found by the rule, from the value of the same sum payable on the extinction of the joint lives AB. Thus, the present value of £100, payable on the extinction of the joint lives AB, is, by Question XXVII., equal to 41·67; whence, if we subtract 14·46 from such value, the difference, or 27·21, will be the present value of £100 payable on the decease of B, provided A be then alive.

These values are, in each case, the sums that ought to be given in a *single* payment; but, if we wish to determine the value of the same in *annual* payments, we must divide those sums by unity added to the value of an annuity on the two joint lives AB. Therefore 14·46 being divided by 13·545 will give 1·068, or £1, 1s. 4d., for the annual payments in the former case, and 27·21 divided by 13·545 will give 2·009, or £2, 0s. 2d., for the annual payments in the latter case.

Example 2. B, aged 60, will, if he lives till the decease of A, aged 25, be entitled to a legacy of £100; what is the value of his interest in such sum, taking the probabilities of living as at *Northampton*, and the rate of interest 5 per cent.?

The value of an annuity on two joint lives OB, aged 26 and 60, is equal 7·365; which being added to unity and multiplied by 4685 (or the number of persons living at the age of 26, as in Table VII.) will give 39190·025; and this, divided by 1·05, will give 37323·833 for the quotient, to be reserved. The value of an annuity on the two joint lives YB, aged 24 and 60, is equal to 7·399; which being multiplied by 4835 (or the number of persons living against the age of 24) will produce 35774·165; this being subtracted from the reserved quotient, and the remainder divided by 4760 (or the number of persons living against the age of 25), will give ·32556. Now, the present value of £1 payable on the extinction of two joint lives aged 25 and 60 is equal to ·60081; therefore ·32556, being subtracted from this value, will leave ·27525; which being multiplied by 50 will give 13·762, or £13, 15s. 3d., for the present value of B's interest in the legacy.

If this sum be subtracted from 60·081 (or the present value of £100 payable on the extinction of the joint lives AB, aged 25 and 60), the difference, or 46·319, will be the present value of the legacy payable on the death of B, provided A be then alive.

And either of these values, divided by unity added to the value of an annuity on the two joint lives, will show the annual payment which ought to be given by B or A respectively, in order to have the same assured to his heirs, provided he dies before the other.

SCHOLIUM.

§ 423. The examples above given show the proper method of proceeding in all similar cases; and it may be here useful to remark that the values adopted by *all* the assurance offices in London[1] have been computed from an incorrect rule given by Mr. Simpson,[2] and therefore cannot be depended upon when the life of A is very young, or when there is any considerable difference between ages of the two lives.

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[1] The values here alluded to have since been altered by the London companies, and for this reason the Table LIII. in the original edition has been omitted here.—EDITOR.

[2] In the *Supplement* to his *Doctrine of Annuities*, Prob. 32, and in his *Select Exercises*, Prob. 32. In using which rule, it should be observed that, when the reversion is a *sum* and not an *estate*, the value found by the rule must be divided by £1 increased by its interest for a year, as explained in page 104. Agreeably to this correction, it will be found that Mr. Simpson's rule may be expressed by the formula $s \times \frac{1 - \rho BB}{2(+1\delta)} \times \frac{\epsilon}{e}$; which denotes half the value of an assurance of the given sum payable on the extinction of two joint lives of the same age with the oldest of the two lives, multiplied by a fraction whose numerator is the *expectation of the life* B, and whose denominator is the *expectation of the life* A.

This is the approximate value when B, or the life in expectation, is the oldest of the two lives. But if B be the youngest, this value must be subtracted from $s \times \frac{1 - \rho AB}{(1+\rho)}$; and the difference will be the value in this case.

I would here observe that Mr. Dodson's formula (*Mat. Rep.*, vol. iii. Ques. 23) is deduced from precisely the *same* series as Mr. Simpson's formula, nevertheless, they give *different* results when expounded numerically.

QUESTION XXXI.

§ 424. **To find the value of a** ***Temporary assurance*** **of a given sum payable on the decease of A, provided he dies** ***before*** **another life B.**

SOLUTION.[1]

Add 2 to the rate of interest; multiply this sum by the value of an annuity on the life B, and add unity to the product. Call this the *first* value.

Add 2 to the rate of interest; multiply this sum by the value of an annuity on a life older than B by the given term, add unity to the product, and then multiply this sum by the expectation of B's receiving £1 at the end of the term. Call this product the *second* value.

Divide the probability that A will die before the end of the term, by the number of years, and multiply the quotient by half the given sum. Call this product the *third* value.

Subtract the *second* value from the *first*, and divide the remainder by the amount of £1 in a year; the quotient thence arising, being multiplied by the *third* value, will give the present value of the given sum required.

Example 1. What is the present value of £100 payable on the decease of A, aged 7, provided that shall happen within fourteen years, and provided another life B, aged 30, be then alive; interest at 4 per cent., and the probabilities of living as observed at *Northampton*?

The value of an annuity on the life B, aged 30, is, by Table X., equal to 14·781; which being multiplied by 2·04 (or 2 added to ·04) will produce 30·15324; and this being added to unity will give 31·15324 for the *first* value.

The value of an annuity on a life fourteen years older than B (that is, on a life aged 44) is equal to 12·472, which being also multiplied by 2·04 will produce 25·44288. This being added to unity will give 26·44288; which being multiplied by ·43804 (or the expectation of B's receiving £1 at the end of fourteen years[2]) will produce 11·58304 for the *second* value.

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[1] It may be necessary to observe that this rule is only an *approximation* to the true value, agreeably to the principles laid down in § 235; and therefore must be always used, not only with caution, but with a due regard to the tables of observation employed. The correct value may be obtained by the help of the formula in Prob. XXVII. cor. 4; but as that formula could not be conveniently expressed in words at length, I have preferred the one above alluded to for the illustration of this part of the subject.

[2] The probability that B shall live to the end of fourteen years is, by the rule in Question I., equal to $\frac{3324}{4385}$; and the present value of £1 certain to be received at the end of that period is, by Table III., equal to ·57748; the product of these two quantities will give ·43804 for the expectation required.

The probability that A will die before the end of the given term is equal to ·14599,[1] which being divided by 14 will give ·010428; and this quotient multiplied by 50 (or half the given sum) will produce ·5214 for the *third* value.

The difference between the *first* and *second* value is 19·5702; which being divided by 1·04 will give 18·8175. This quotient multiplied by ·5214 will produce 9·8114 for the value required.

This is the sum that ought to be given for the assurance in a *single* payment; but if we wish to determine the value of the same in annual payments, we must divide this sum by 9·566 (or unity added to the value of an annuity on the two joint lives for thirteen years); which will give 1·094 for the annual payment required.

Example 2. B, aged 60, will, if he lives to the decease of A, aged 25, be entitled to a legacy of £100, provided that event shall happen within fifteen years; what is the value of his interest therein, reckoning the probabilities of life as observed at *Northampton*, and the rate of interest at 5 per cent.?

The value of an annuity on the life aged 60 is, by Table X., equal to 8·392, which being multiplied by 2·05, and added to unity, will make 18·2036 for the *first* value.

The value of an annuity on a life aged 75 is equal to 4·744, which being multiplied also by 2·05, and added to unity, will make 10·7252; and this being multiplied by ·19637 (or the expectation of B's receiving £1 at the end of fifteen years[2]) will produce 2·1061 for the *second* value.

The probability that A will die before the end of fifteen years is equal to ·23634;[3] which being divided by 15 will give ·015756; and this quotient, multiplied by half the given sum, will produce ·7878 for the *third* value.

The difference between the *first* and *second* value is 16·0975; which being divided by 1·05 will give 15·3310. This quotient, multiplied by ·7878, will produce 12·078 for the value required, in a *single* payment.

And this sum being divided by 7·592 (or unity added to the value of an annuity on the two joint lives for fourteen years) would give 1·591 for the value of the same sum, if required, in *annual* payments.

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[1] The probability that A shall *live* to the end of fourteen years is, by the rule in Question I., equal to [illegible]; that is, equal to ·85401. This value, subtracted from unity, will give ·14599 for the probability that A shall *die* before the end of that period.

[2] The probability that B shall live to the end of fifteen years is equal to $\frac{832}{2038}$, and the present value of £1 to be received at the end of that term is equal to ·48102; the product of these two quantities will give ·19637 for the expectation required.

[3] The probability that A will die in fifteen years is equal to $\frac{3635}{4760}$ subtracted from unity; that is, equal to $\frac{1125}{4760}$, or ·23634.

SCHOLIUM.

§ 425. If the term for which the assurance is made happens to fall within the limits of equal decrements, of the life A, as found in the given table of observations, it is obvious (from the method of deduction in § 235, &c.) that this rule will give the *exact* value. This is the case in the second example here given; for, by referring to Table VII., it will be found that from the age of 25 to 40 the decrements of life are exactly equal, and consequently the rule is in this case strictly correct. Nevertheless, if the value of the same assurance be found by the help of the formula given in Prob. XXVII. cor. 4, it will come out equal to 12·139, and I can account for this difference in no other way than by supposing there is some error in the tables of the values of the annuities, for it is evident that the two results *ought* to be precisely the same.

QUESTION XXXII.

§ 426. To find the value of an *Assurance* of a given sum payable on the decease of A, provided he dies *after* another life B.

SOLUTION.

From the value of the assurance of the given sum payable on the decease of A, subtract the value of the same assurance payable on the decease of A, provided he dies *before* B; the difference will be the value required. See § 241.

Example 1. What is the present value of £100 payable on the decease of A, aged 20, provided B, aged 40, be then dead, reckoning interest at $4\frac{1}{2}$ per cent., and the probabilities of living as observed by *M. De Parcieux?*

The value of the assurance of £100, payable on the decease of A, is, by Question XXVII., equal to 24·107; and the value of the same sum, payable on the decease of A, provided he dies before B, is, by Question XXX., equal to 14·460; consequently 9·647, or £9, 12s. 11d., will be the value required.

Having thus found the value of the given sum payable on the decease of A, provided B be then dead, we may easily determine the value of a similar sum payable on the decease of B, provided A be the extinct; for, we have only in such case to deduct the value, found by the rule, from the value of the same sum payable on the extinction of the longest of the two lives. Thus, the present value of £100 payable on the extinction of the longest of two lives aged 20 and 40 (at the rate of interest, &c., above mentioned) is, by Question XXVII., equal to 16·748; from which, if we subtract 9·647, as above found, the difference, or 7·101, will be the value of £100 payable on the decease of B, provided he dies after A.

These sums are the values which ought in each case to be given in a *single* payment; but if we wish to determine the value of the same in *annual* payments till the claim *is determined*, we must divide the single payment thus found, by unity added to the value of an annuity on the two joint lives. Or, if we wish to determine the value of the same in *annual* payments till the claim *becomes due*, we must divide the single payment by unity added to the value of an annuity on the single life, on which the assurance is made.[1] Thus, 9·647 being the value, in a *single* payment, of an assurance of £100 on the life A, provided he dies after B; it follows that the value of the same assurance, in *annual* payments till the claim *is determined*, is equal to 9·647 divided by 13·545; that is, equal to ·712, or 14s. 3d. And that the value of the same assurance, in *annual* payments till the claim *becomes due*, is equal to 9·647 divided by 17·624; that is, equal to ·547, or 10s. 11d.

Example 2. What is the present value of £100 payable on the decease of A, aged 25, provided he should die after B, aged 60; interest at 5 per cent., and the probabilities of living as at *Northampton?*

The value of an assurance of £100 on the decease of A is, by the rule in Question XXVII., equal to 30·633; and the value of the same sum payable at the same period, provided B be then extinct, is, by Question XXX., equal to 13·762; consequently 16·871, or £16, 17s. 5d. will be the value required in a *single* payment.

This value, being divided by 8·383 will give 2·013, or £2, 0s. 3d., for the *annual* payments till the claim *is determined;* or, being divided by 14·567, will give 1·158, or £1, 3s. 2d., for the *annual* payments till the claim *become due.*

QUESTION XXXIII.

§ 427. To find the value of a *Temporary assurance* of a given sum payable on the decease of A, provided he dies *after* another life B.

SOLUTION.

From the value of a temporary assurance of the given sum payable on the decease of A, subtract the value of a similar temporary assurance, payable on the decease of A, provided he dies *before* B; the difference will be the value required.

Example. What is the present value of a legacy of £100 payable on the decease of A, aged 25, provided he dies within fifteen years, and provided also that another person B, now aged 60, be then dead; interest being reckoned at 5 per cent., and the probabilities of living as at *Northampton?*

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[1] See p. 208.

The value of an assurance of £100 payable on the decease of a person aged 25, provided that should happen within fifteen years, is, by the rule in Question XXIX., equal to 16·354;[1] and the value of a similar assurance, provided another person now aged 60 be then living, is, by Question XXXI., equal to 12·078; consequently this latter value, subtracted from the former, will leave 4·276, or £4, 5s. 6d., for the value of the assurance required.

SCHOLIUM.

§ 428. This rule will still be correct although the given term *exceed* the number of years between the age of B, and the oldest life in the table of observations; but in such case the value of the assurance payable on the decease of A, provided he dies *before* B (being now for the *whole continuance* of the joint lives) must be found by Question XXX. instead of Question XXXI.; and the value thus found, being deducted from the temporary assurance on the life A, will give the value required.[2]

Thus, if the term in the preceding example had been forty years, then the value of the assurance on the life A for forty years would, by the rule in Question XXIX., be equal to 27·682;[3] and the value of the assurance on the same life, provided B be alive at his decease, would, by Question XXX., be equal to 13·762; consequently this latter value, subtracted from the former, would leave 13·920, or £13, 18s. 5d., for the value required in a *single* payment. And 13·920 divided by 8·383 (or unity added to the value of an annuity on the two joint lives) will give 1·661 for the value of the same in *annual* payments till the *claim* is determined; or, being divided by 14·164 (or unity added to the value of an annuity on the life A for thirty-nine years) will give ·983 for the same value in *annual* payments till the sum *becomes due*.

§ 429. Before I close the present chapter, I shall insert the solution of a question, which will be often found of considerable practical utility, not only to individuals, but likewise to those Societies whose business consist in granting assurances on lives.

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[1] The value of the assurance on the *whole continuance* of the life is, by the rule in Question XXVII., equal to 30·633; and the value of the same assurance *deferred* for fifteen years is, by the rule in Question XXVIII., equal to 14·279; consequently the difference between these two values will be the value of the *temporary* assurance.

[2] Mr. Morgan has given a singular and troublesome rule for this case, in Dr. Price's *Obs. on Rev. Pay.*, vol. i. p. 70, note i. It is neither simple nor *correct.*

[3] The value of the assurance on the *whole continuance* of the life is, as in the preceding note, equal to 30·633; and the value of the same assurance *deferred* for forty years is, by the rule in Question XXVIII., equal to 2·951; consequently the latter value subtracted from the former will be the value of the *temporary* assurance required.

The reader must be already aware that if a person were to make an assurance at any of the offices on his own life for a *single* year, and to repeat this at the end of every successive year to the utmost extremity of life, the annual payment (for such assurance) would be continually *increasing* till his death. But, if he made the assurance on the *whole continuance* of his life, and contracted with the office to pay the value of such assurance by *equal* annual payments during his life (as is usually the case), it is evident that such annual payment ought to be *greater* than the premium required for an assurance for a single year at his *present age*, but *less* than the premium required for a similar assurance at the more *advanced periods* of life. Hence, it appears that if a person, who was originally assured for the whole term of his life, should be desirous (either through inability, or any other motive) of renouncing his claim upon the office and of cancelling his policy, he ought to have some part of those annual payments returned to him, or, in other words, a compensation ought to be made him for that *excess* in the annual payments which he has been advancing to the Society. The object of the following question is to determine the amount of that remuneration.

QUESTION XXXIV.

§ 430. To find the sum that ought to be given to a person, who is assured for the whole term of his life for a given sum, in order that he may *renounce his claim* thereto.

SOLUTION.

Multiply the equal annual payment, which he has been giving since the assurance commenced, by the value of an annuity (increased by unity)[1] on the life at the present age; subtract the product from the value of the assurance of the given sum on the life at its present age; the difference will be the sum required.

Example. A person *now* aged 50, who has been paying 21·790, or £21, 15s. 10d.,[2] annually for the assurance of £1000 at his death, is desirous of discontinuing the same; what sum ought to be given to him by the office as a compensation for so doing; interest being reckoned at 3 per cent., and the probabilities of living as at *Northampton?*[1]

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[1] This supposes that the policy is cancelled immediately *before* the annual payment becomes due; but if immediately *after*, we must multiply the payment, above alluded to, by the value of an annuity on the given life without the addition of unity.

[2] This is the annual payment for the assurance of £1000 on a life aged 20.

The value of an annuity (increased by unity) on a life aged 50 is, by Table X., equal to 12·264; which being multiplied by 21·790, will produce 267·233; this being subtracted from 608·660 (or the value of the assurance on a life aged 50, as appears by Table LI.) will leave 315·890, or £315, 17s. 10d., for the answer required.[2]

SCHOLIUM.

§ 431. If the sum which is to be received at the death of the assured has been increased by any addition or bonus (as will oftentimes be the case in the *Equitable Society*), we must subtract the product above alluded to from the value of the assurance of the given sum, *together with its bonus;* and the remainder will, in this case, be the sum required.[3]

Example. Suppose that the several additions made to the policy, alluded to in the preceding example, amount to £1800,[4] in which case the executors of the assured would be entitled to £2800 on his death; what sum ought now to be given him for renouncing his claim upon the Society; the interest, &c., being as in the preceding example?

The value of an assurance of £2800 on a life aged 50 is 1704·248, from which we must subtract 292·770; the difference is 1411·478, or £1411, 9s. 7d., which is the answer required.

The same result would have been obtained by adding £1095, 11s. 9d. (or the value of an assurance of the additional £1800 on the given life) to £315, 17s. 10d., as found by the preceding example.

§ 432. These two examples will show the method of proceeding in all similar cases, whether the assurance depends on a single life, upon any joint lives, or on any other contingency. It will also serve to show the amount of the *debts* owing by a company, whose business consists in making assurances on lives, since these are the sums which would be required to cancel the respective claims on the Society, and may consequently be fairly considered as money owing by them. These debts, therefore, being deducted from the amount of capital in hand, will leave the net *surplus* stock belonging to the Society; and it is this net surplus stock *alone* that can be considered as the true *profits* of the concern, and as the *only* fund from which any divisions ought to be made amongst the different members, either by way of interest, dividend, or bonus. A Society that is not guided by some principle of this kind must inevitably terminate in disgrace and ruin.

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[1] The rate of interest and probabilities of life, in *such* computations, *ought* to be the *same* as those adopted by the office at which the policy is effected.

[2] The truth of this rule will be evident from the following statement:—The Society may be considered as indebted to the assured in the present value of an assurance of £1000 on a life aged 50; which is equal to 608·66, or £608, 13s. 3d. And the assured may be considered as owing to the Society the present value of all the annual payments of £21, 15s. 10d. during the remainder of his life; the first of which payments is supposed to be made immediately; therefore the value of all those payments will be equal to 21·790 multiplied by 13·436; which produces 292·77, or £292, 15s. 5d. Consequently the interest of the assured in his policy will be equal to the *difference* between £608, 13s. 3d., and £292, 15s. 5d.; that is, equal to £315, 17s. 10d., as found by the example in the text.

[3] Or, we may add the value of an assurance of the *additional* sum on the given life to the value found by the preceding solution, which will give the same results.

[4] This would actually be the case at the *Equitable Society*. So that this person, although originally assured there for *one thousand* pounds only, would now be entitled to receive above *one thousand four hundred* pounds for *cancelling his policy!*

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CHAPTER XIII.

ON SCHEMES FOR PROVIDING ANNUITIES FOR THE BENEFIT OF OLD AGE, AND OF WIDOWS.

1. *For Old Age.*

§ 433. The rule given for the solution of Question VI., in the preceding Chapter, is of considerable practical use, since by means of it we are enabled to determine the efficacy and propriety of those schemes and establishments which are proposed with a view of providing *Annuities for Old Age*. For, having found the present value of an annuity of £1 per annum on any given life or lives, to commence at the end of any number of years from the present time, we may easily find the present value of any *other* annuity (either in a *single* payment, or in *annual* payments, or in *both*) by multiplying the present values, thus deduced, by that other annuity, agreeably to the principles already laid down in the note in page 214. And in like manner, having found the present value above alluded to, we may readily determine the annuity that ought to be given, at the end of any period, for a given sum paid down immediately, or for a given sum, *part* of which is paid immediately, and the *remainder* by annual payments till the end of the given term.

Thus it appears, by the second example in page 222, that the present value of an annuity of *one pound* per annum on the life of a female aged 40, to commence at the end of ten years, is (reckoning interest at 4 per cent., and the probabilities of life as observed in *Sweden*) equal to 6·926, or £6, 18s. 6d., consequently an annuity of £44 per annum, to commence at the same period, would be equal to 304·744, or £304, 14s. 11d., in a *single present* payment. And this latter sum divided by 7·900 (or unity added to the value of an annuity on the given life for nine years) will give 38·575, or £38, 11s. 6d., for the value of the same annuity in *annual* payments: the first of such payments to be made immediately, and the remaining ones at the beginning of every subsequent year, provided the given life is then in being. Or, if £200 (part of the £304, 14s. 11d.) be paid down *immediately*, then the *annual* payments which ought in this case to be made, in order to supply the deficiency, will be equal to the remaining sum of 104·744 divided by 7·900;[1] that is, equal to 13·259, or £13, 5s. 2d.

In like manner, if we wish to determine the *annuity* that ought to be granted on such life at the end of ten years, for £73, 10s., paid down *immediately*, and £6, 14s. in *annual* payments, we must proceed as follows: Multiply 6·700 (or the amount of the annual payment) by 7·900 (or unity added to the value of annuity on the given life for nine years),[2] which will produce 52·930, or £52, 18s. 7d., for the total present value of such annual payments; and which being added to the admission money, or £73, 10s., will make the whole value given for the annuity equal to 126·430, or £126, 8s. 7d. Therefore this quantity, being divided by 6·926 (agreeably to the principles laid down in the note in page 215), will give 18·254, or £18, 5s., for the amount of the annuity required.

§ 434. This example will show the method of proceeding in all similar cases; and I have taken this one in particular, because it is the one assumed by Dr. Price (*Obs. on Rev. Pay.*, vol. i. p. 137) to expose the futility and iniquity of those schemes that were published by the several societies instituted about the year 1770, who by holding out a false lure to the public, took in the unwary, and entailed misery and distress on the unfortunate adventurers.[3] Happily his efforts were crowned with success, and none of those public societies now exist to disgrace the present age. But although he has sufficiently demonstrated that those bubbles could not possibly comply with the terms held out to the public, yet he has not shown that they were so deficient and absurd as they will appear to be, when compared with the values deduced as above from the real probabilities of life.[1] The present examples will be sufficient to enable any one to examine the accuracy and sufficiency of similar proposals; and I shall not detain the reader with any further comments on this subject, as it is probable that a similar circumstance will not soon again occur to call forth the censure of every honest member of society. The public are now better acquainted with the method of calculating the values that ought to be

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[1] If the first of these annual payments is not made till the *end* of the first year, then the sum here alluded to must be divided by 7·475 (or the value of an annuity on the given life for ten years) which will make the annual payment equal to 14·013, or £14, 0s. 3d.; and in this case the last annual payment will be made at the same time that the first payment of the annuity becomes due. In the cases mentioned by Dr. Price (*Obs. on Rev. Pay.*, vol. i. p. 141), and to which I shall presently allude, the payments were made *half*-yearly, and the first of those half-yearly payments was paid down with the admission money; but he does not notice this fact, although it makes some difference in the results.

[2] If the first annual payment is not to be made till the *end* of the year, we must multiply 6·700 by 7·475 (or the value of an annuity on the given life for ten years). See the preceding note.

[3] The following are the terms upon which some of those societies granted annuities to

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members aged 40, for what might happen to remain of their lives after 50 years of age:—

	Admission Money.	Annual Payment.	Annuity Granted.
The Laudable Society, . . .	£73 10 0	£6 14 0	£44 0 0
The London Annuitants, . . .	25 0 0	10 0 0	44 0 0
The Equitable Society of Annuitants,	32 0 0	13 0 0	44 0 0
The London Union Society, . .	37 0 0	7 0 0	54 12 0
The Amicable Society of Annuitants,	28 16 0	6 0 0	26 0 0
The Provident Society, . . .	31 10 0	8 8 0	25 0 0

The annual payment was usually paid *half*-yearly; and the *first* half-yearly payment was made at the time of admission. It will be readily seen (from the example given in the text) that there was no probability that the Societies would ever be able to *continue* these enormous annuities to *all* the members. The truth is, that (even at that day) they were styled *impositions on the public, proceeding from ignorance, and supported by credulity and folly.* "But," as Dr. Price justly continues, "this is too gentle a censure. There is reason to believe that *worse* principles have contributed to their rise and support. The present members, consisting chiefly of persons in the more advanced ages, who have been admitted on the easiest terms, believe that the schemes they are supporting will last their time and that they will be gainers. And, as to the injury that may be done to their successors or to younger members, is at a distance, and they care little about it. Agreeably to this principle, the founders of these societies begin so low as not to require perhaps a *fourth* or a *fifth* of the values of the annuities they promise. Afterwards they advance gradually, just as if they imagined that the value of the annuities was nothing determinate, but increased with every increase of the society. But, as no ignorance can believe this, the true design appears to be, to form soon as large a society as possible, by leading the unwary to endeavour to be *foremost* in their applications, lest the advantage of getting in on the easiest terms should be lost. It is well known that these arts have succeeded wonderfully; and that, in consequence of them, these societies now consist of persons who for the *same* annuities make higher or lower payments, according to the time when they have been admitted, and the generality of whom, therefore, must know that either more than the values have been required of the members last admitted, or, if not, that they are themselves expecting considerable annuities for which they have given no valuable consideration, and which, if paid, must be *stolen* from the pockets of some of their fellow-members."

[1] Dr. Price formed his calculations from the values of annuities deduced from the hypothesis of *M. De Moivre.*

given in such cases, and are not likely to become again in *this* manner the dupes of any artful impostor. Their danger now seems to lie in the *opposite* extreme.

I do not, however, by these observations at all intend to discourage institutions or schemes for providing annuities for old age. On the contrary, I am fully persuaded, that a society or office that would proceed upon an efficient or *liberal* plan might be of essential advantage to the state and highly beneficial to the public. Many persons, particularly in the inferior stations of life, would in such case be induced to lay by (during the period of youth and vigour) many small sums, which are now squandered in riot and dissipation; and by endeavouring to get a little money beforehand, would acquire habits of industry, and be probably enabled thus to make provision for themselves in the more advanced periods of life, when they will be incapable of labour, thereby rendering themselves not only more independent and valuable members of society, but also obviating the necessity of their applying to the parish for relief. Dr. Price justly observes, that " the lower orders of mankind are objects of particular compassion, when rendered incapable, by accident, sickness, or age, of earning their subsistence. This has given rise to many very useful societies among them for granting relief to one another, out of little funds supplied by weekly contributions." It is much to be feared, however, that many of these establishments are formed on such vague and inefficient plans, without any regard to the true principles of calculation, that they are not always entitled to our unqualified approbation. When thus erroneously founded, and perhaps at the same time badly and carelessly conducted, they only serve to increase the misery which they were intended to remove.

§ 435. Under these circumstances it is to be regretted that the Legislature has not adopted some efficient measures towards assisting such persons, in the humbler stations of life, as might be desirous of employing their money in this laudable manner. Two attempts have been made to obtain an Act for this purpose; but, though it has each time passed the House of Commons, it has been as regularly thrown out by the House of Lords. I cannot here, however, omit the opportunity of inserting the pertinent remarks on this subject of a very able writer,[1] which will serve to show the objects that those bills had in view.

" To make such a provision for one's old age, is so natural a piece of prudence, that it seems wonderful at first sight that it should not be generally practised by the labouring poor above described, as it is almost universally by persons engaged in the higher paths of industry, such as the several branches of commerce, and the practice of the liberal professions; nor can their negligence in this respect be accounted for in any other way so naturally as by ascribing it to their wanting opportunities of employing the money they might save in some safe and easy method that would procure them a suitable advantage from it in the latter periods of their lives. They know, for the most part, but little of the public funds; and when it happens that they are acquainted with them, the smallness of the sums they would be entitled to receive (as the interest of the money they could afford to lay out in them) is no encouragement to dispose of it in that way. What inducement, for instance, can it be to a man, who has saved ten pounds out of his year's wages, to invest it in the purchase of three per cent Bank annuities, to consider that it will produce him about six or seven shillings a year? It is but the wages of three days' labour. And if they lend their money to tradesmen of their acquaintance, as they sometimes do, it happens not unfrequently that their creditor becomes a bankrupt, and the money they had trusted him with is lost for ever; which discourages others of them from saving their money at all, and makes them resolve to spend it in the enjoyment of present pleasure. But if they saw an easy method of employing the money they could spare, in such a manner as would procure them a considerable income in return for it at some future period of their lives, without any such hazard of losing it, by another man's folly or misfortune, it is probable they would frequently embrace it; and thus a diminution of the poor's rate on the estate of the rich, an increase of the present industry and sobriety in the poor, and a more independent and comfortable support of them in their old age than they can otherwise expect, would be the happy consequences of such an establishment. Now this, I conceive, might be effected in the following plain and easy method:—

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[1] Mr. Baron Maseres, in a pamphlet which he published, entitled *A Proposal for Establishing Life Annuities in Parishes for the Benefit of the Industrious Poor*: London, 1772. The learned author has lately favoured me with a copy of that pamphlet, with an obliging offer that I might make what use of it I pleased in my intended publication; and I lament that the limits of the present treatise will not enable me to enter upon the subject more than by an insertion of the above extract from that valuable work.

"1. Let the churchwardens and overseers of every parish be empowered by Act of Parliament to grant life annuities to such of the inhabitants of the parish as shall be inclined to purchase them, to commence at the end of one, two, or three years (or such other future period of time as the purchaser shall choose) and to be paid out of the poor's rate of the parish, so that the lands and other property in the parish that is chargeable to the poor's rate, shall be answerable for the payment of these annuities. This circumstance would give these annuities great credit with the poor inhabitants, by setting before them so solid and ample a security for the regular payment of them.

"2. Let the annuities, thus granted to the poor inhabitants, be such as arise from a supposition that the interest of money is three per cent.; or some higher rate of interest, if the churchwardens or overseers of the poor shall think fit to make use of such higher interest.

" 3. But, at the rate of three per cent., the purchaser should have a right to an annuity, and the churchwardens and overseers of the poor should be compellable to grant it.

" 4. No annuity, depending upon one life, should exceed £20 a year.

" 5. No less sum than £5 should be allowed to be employed in the purchase of an annuity. This is to avoid intricacy and multiplicity of accounts.

" 6. An exact register of these grants should be kept by the churchwardens and overseers of the poor in proper books for the purpose, in which the grants should be copied exactly, and the copy of each grant subscribed by the person to whom it is granted; or, if he cannot write, marked with his mark, and subscribed by two other persons as witnesses. And this copy in the register-book of the parish should be good evidence of the purchaser's right to the annuity, in case the original deed of grant to the purchaser (which was delivered to him at the time of the purchase) should afterwards be lost.

" 7. The money thus paid to the churchwardens and overseers of the poor for the purchase of life annuities should be employed in the purchase of three per cent Bank annuities, in the joint names of all the churchwardens and overseers; and by them transferred, at the expiration of their offices, to their successors, and so on to the next successors for ever, so as to be always the legal property of the churchwardens and overseers of the poor for the time being, in trust for the persons who should be entitled to the several life annuities, granted in the manner above mentioned; and the interest of this money should be received every half-year, and invested in the purchase of more principal annually, so as to make a perpetual fund for the payment of these life annuities. And when any annuity became due, the churchwardens and overseers of the poor should pay it out of this fund, and should sell a sufficient part of the principal for that purpose, where the interest was not sufficient for the purpose, as would generally be the case; and the deficiencies (if any were) of both principal and interest should be made good out of the poor's rate assessed upon the parish. But this could hardly ever happen, if the annuities granted to the purchasers were such as would be proportional to the money contributed, upon a supposition that the interest of money was only three per cent., because that is a lower interest than that which the parish would receive from the Government for the same money, by investing it in the three per cent Bank annuities, as that stock is now twelve per cent. under par,[1] and is not likely soon to rise to par again. So that the landholders of the parish, and the owners of other rateable property in it, need be under very little apprehension of having their estates exorbitantly burthened by a great increase of the poor's rate, in order to make good the payment of these annuities. On the contrary, they would be gainers by this institution, as was observed above, since many of the poor who must otherwise, in their old age, come to be a burthen upon the parish, would now be maintained, in part at least, by annuities paid to them out of a fund of their own raising."

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[1] It is now (1810) above 30 per cent. under par.

Agreeably to this plan, a bill[1] was brought into the House of Commons in the year 1773 by Mr. William Dowdeswell, and passed, on a division after a debate, by a majority of two to one of all the members present; but it was rejected by the House of Lords in consequence of the opposition of Lord Cambden, who conceived that the measure might be ultimately injurious to the *landed interest;* since the value of future leases might be affected by an increase of the poor's rate to make good any deficiency arising from the failure or defection of the scheme. A bill of a similar nature, however, with tables computed by Dr. Price,[2] was introduced in the year 1789, but shared a similar fate.

2. *For Widows.*

§ 436. The rules by which Questions XIII. and XVIII., in the preceding chapter, are solved, are very useful in enabling us to determine the advantage and propriety of those schemes which are instituted with a view to provide *Annuities for Widows.*[3] For, having found the present value of a reversionary annuity, or a deferred reversionary annuity, of £1 per annum on any given life after any other given life, we may readily find the present value of any other annuity (either in a *single* payment, or in *annual* payments, or in *both*) by multiplying the present value, thus obtained, by that other annuity. And, in like manner, having found the present value above alluded to, we may easily determine the annuity which ought to be granted, to the life in reversion, for a given sum paid down immediately; or for a given sum, *part* of which is paid down immediately, and the *remainder* by annual payments during the existence of the life in possession. Or, according to any other plan which may be proposed.

Thus, let the scheme of a society for granting annuities to widows be such that, if a member lives *one* year after admission, his widow shall be entitled to a life annuity of £20; if *seven* years, to £10 more (or £30 in the whole); if *fifteen* years, to £10 more (or £40 in the whole). What *ought* to be the annual payments for members aged 30, 40, and 50 years respectively, and supposing them of the same ages as their wives; interest being reckoned at 4 per cent., and the probabilities of living as observed in *Sweden* amongst males and females respectively; and also as observed at *Northampton* and at *London?*

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[1] A copy of this bill, together with the tables that were computed for it, are inserted by Mr. Baron Maseres in his *Doctrine of Life Annuities.*

[2] These tables are inserted in Dr. Price's *Obs. on Rev. Pay.*, vol. ii. p. 473.

[3] These observations relate to the method of determining the best mode of providing *Annuities* for Widows; but those inquiries which relate to the best mode of providing for the payment of any given *Sum* to a widow, on the death of her husband, are answered by Question XXX. in the preceding chapter.

By proceeding according to the rules laid down in the Scholium in § 394,[1] we shall find that the annual payments which ought to be made by the members, of the respective ages above mentioned, will (according to the several tables of observations made use of) be as follow:—

Ages.		Sweden.	Northampton.	London.
30	=	6·90	7·66	8·52
40	=	7·89	8·09	9·06
50	=	8·50	8·40	9·51

I have selected this scheme as being that on which the *London Annuity Society for the Benefit of Widows* set out in 1765. The *Laudable Society*, which was formed on nearly a similar plan, had been established in the year 1761. In each of these, the *annual* contribution of every member was five guineas only, payable half-yearly; for which payment, his widow was entitled to the annuities stated in the scheme above mentioned, according to the conditions there expressed. Nothing, however, can more fully show the inadequacy of the means for carrying into effect the intentions of those Societies than the examples above given. For it will be seen that, on the supposition that the members are of the *same ages* as their wives, the Societies received, on an average, little more than *three-fifths* of the true value of the annuities; but on the supposition that they are, one with another, *ten years older* than their wives, it will be found that they received only *one half* of the true values of such annuities. The consequences of such inequitable measures were highly injurious. The more early annuitants enjoyed the *full* sums, according to the conditions of the plan; but, after pursuing this pernicious course for some years, the directors at length listened to the voice of reason; and, in consequence of the repeated warnings that were given them, found it necessary to adopt one of the two following plans in order to maintain the permanency and security of the establishment:—to *increase* the premiums, or *diminish* the annuity. The *London Annuity Society*, at an early period, adopted the former proposal, and thus preserved its honour and its credit; but the *Laudable Society*, although it repeatedly reduced the annuities, had been too long struggling with the errors of its original establishment to enable it to derive much permanent advantage from this procrastinated remedy. "If circumstances, therefore, should still continue unfavourable, the next measure must be the dissolution of the Society, and a division of the remaining capital among the annuitants and surviving members, in proportion to their respective interests in the funds of the Society."[1]

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[1] For example: the value of a reversionary annuity on the life of a female aged 30, after the life of a man aged also 30, and deferred for *one* year, provided *both* the lives continue so long, is (by the rule in the Scholium, according to the *Sweden* observations) equal to 3·108; consequently the present value of an annuity of £20, under the same circumstances, is 62·160. In like manner, the present value of £10 per annum on the same life, deferred for *seven* years, provided *both* the lives continue so long, is equal to 21·680. And by the same rule the present value of £10 per annum, deferred for *fifteen* years, is under the like circumstances equal to 12·566.

These values, being added together, are equal to 96·404; or the value of the expectation, described in the scheme, in a *single* present payment, and which being divided by 13·965 (or unity added to the value of an annuity on the two joint lives) will give 6·903 for the value of the same expectation in *annual* payments, as stated in the text. In like manner may be found the answer to all questions of a similar kind.

§ 437. Such will be the final issue of every scheme that is not founded on correct observations and on mathematical principles; and though my remarks have been confined to the two establishments above mentioned, they will, I fear, apply with too much justice to several others of a similar nature. "There are in this kingdom many institutions for the benefit of widows besides the two on which I have now remarked; and in general, as far as I have had any information respecting them, they are founded on plans equally inadequate, having been formed just as fancy has dictated, without any knowledge of the principles on which the values of reversionary annuities ought to be calculated. The motives which influence the contrivers of these institutions may be *laudable;* but they *ought*, I think, to have informed themselves better." "The longer such schemes are carried on, the more mischief they must produce. 'Tis vain to form such establishments with the expectation of seeing their fate determined *soon* by experience. If not more extravagant than any ignorance can well make them, they *will* go on prosperously for twenty or thirty years; and if at all tolerable, they *may* support themselves for forty or fifty years, and at last end in distress and ruin. All inadequate schemes lay the foundation of *present* relief on *future* calamity, and afford assistance to a *few* by disappointing and distressing *multitudes*."[2]

The very learned and able writer, from whose work these quotations are made, employed his great abilities in detecting the pernicious tendency and iniquity of the several schemes above alluded to. His remarks on this subject are invaluable, and will always be consulted with advantage. He has not only shown how far the various societies, then in existence, erred from the true line of equity and propriety (predicting therefrom their incapacity and ruin); but has likewise pointed out some of the *best* schemes for providing annuities for widows, such as might be "durable, and at the same time easy and encouraging." As I cannot add much to the observations of so intelligent an author, I must refer the reader to that work for any further information on this subject.

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[1] See the history of these two Societies brought down to the present period, in Dr. Price's *Obs. on Rev. Pay.*, vol. i. pp. 72—104.

[2] Dr. Price's *Obs. on Rev. Pay.*, vol. i. chap. ii.

TABLES.

Years	3 per Cent.	3½ per Cent.	4 per Cent.	4½ per Cent.	5 per Cent.	Years
1	1·0300000	1·0350000	1·0400000	1·0450000	1·0500000	1
2	1·0609000	1·0712250	1·0816000	1·0920250	1·1025000	2
3	1·0927270	1·1087179	1·1248640	1·1411661	1·1576250	3
4	1·1255088	1·1475230	1·1698586	1·1925186	1·2155063	4
5	1·1592741	1·1876863	1·2166529	1·2461819	1·2762816	5
6	1·1940523	1·2292553	1·2653190	1·3022601	1·3400956	6
7	1·2298739	1·2722793	1·3159318	1·3608618	1·4071004	7
8	1·2667701	1·3168090	1·3685691	1·4221006	1·4774554	8
9	1·3047732	1·3628974	1·4233118	1·4860951	1·5513282	9
10	1·3439164	1·4105988	1·4802423	1·5529694	1·6288946	10
11	1·3842339	1·4599697	1·5394541	1·6228531	1·7103394	11
12	1·4257609	1·5110687	1·6010322	1·6958814	1·7958563	12
13	1·4685337	1·5639561	1·6650735	1·7721961	1·8856491	13
14	1·5125897	1·6186945	1·7316765	1·8519449	1·9799316	14
15	1·5579674	1·6753488	1·8009435	1·9352824	2·0789282	15
16	1·6047064	1·7339860	1·8729813	2·0223702	2·1828746	16
17	1·6528476	1·7946756	1·9479005	2·1133768	2·2920183	17
18	1·7024331	1·8574892	2·0258165	2·2084788	2·4066192	18
19	1·7535061	1·9225013	2·1068492	2·3078603	2·5269502	19
20	1·8061112	1·9897889	2·1911231	2·4117140	2·6532977	20
21	1·8602946	2·0594315	2·2787681	2·5202412	2·7859626	21
22	1·9161034	2·1315116	2·3699188	2·6336520	2·9252607	22
23	1·9734865	2·2061145	2·4647156	2·7521664	3·0715238	23
24	2·0327941	2·2833285	2·5633042	2·8760138	3·2250999	24
25	2·0937779	2·3632449	2·6658363	3·0054345	3·3863549	25
26	2·1565913	2·4459586	2·7724698	3·1406790	3·5556727	26
27	2·2212890	2·5315671	2·8833686	3·2820096	3·7334563	27
28	2·2879277	2·6201720	2·9987033	3·4297000	3·9201291	28
29	2·3565655	2·7118780	3·1186515	3·5840365	4·1161356	29
30	2·4272625	2·8067937	3·2433975	3·7453181	4·3219424	30
31	2·5000804	2·9050315	3·3731334	3·9138575	4·5380395	31
32	2·5750828	3·0067076	3·5080588	4·0899810	4·7649415	32
33	2·6523352	3·1119424	3·6483811	4·2740302	5·0031885	33
34	2·7319053	3·2208603	3·7943163	4·4663615	5·2533480	34
35	2·8138625	3·3335905	3·9460890	4·6673478	5·5160154	35
36	2·8982783	3·4502661	4·1039326	4·8773785	5·7918161	36
37	2·9852267	3·5710254	4·2680899	5·0968605	6·0814069	37
38	3·0747835	3·6960113	4·4388135	5·3262192	6·3854773	38
39	3·1670270	3·8253717	4·6163660	5·5658991	6·7047512	39
40	3·2620378	3·9592597	4·8010206	5·8163645	7·0399887	40
41	3·3598989	4·0978338	4·9930615	6·0781009	7·3919882	41
42	3·4606959	4·2412580	5·1927839	6·3516155	7·7615876	42
43	3·5645168	4·3897020	5·4004953	6·6374382	8·1496669	43
44	3·6714523	4·5433416	5·6165151	6·9361229	8·5571503	44
45	3·7815958	4·7023586	5·8411757	7·2482484	8·9850078	45
46	3·8950437	4·8669411	6·0748227	7·5744196	9·4342582	46
47	4·0118950	5·0372840	6·3178156	7·9152685	9·9059711	47
48	4·1322519	5·2135890	6·5705282	8·2714556	10·4012697	48
49	4·2562194	5·3960646	6·8333494	8·6436711	10·9213331	49
50	4·3839060	5·5849269	7·1066834	9·0326363	11·4673998	50
51	4·5154232	5·7803993	7·3909507	9·4391049	12·0407698	51
52	4·6508859	5·9827133	7·6865887	9·8638646	12·6428083	52

Years	3 per Cent.	3½ per Cent.	4 per Cent.	4½ per Cent.	5 per Cent.	Years
53	4·7904125	6·1921082	7·9940523	10·3077385	13·2749487	53
54	4·9341249	6·4088320	8·3138144	10·7715868	13·9386961	54
55	5·0821486	6·6331411	8·6463669	11·2563082	14·6356309	55
56	5·2346131	6·8653011	8·9922216	11·7628420	15·3674125	56
57	5·3916515	7·1055866	9·3519105	12·2921699	16·1357831	57
58	5·5534010	7·3542822	9·7259869	12·8453176	16·9425722	58
59	5·7200030	7.6116820	10·1150264	13·4233569	17·7897009	59
60	5·8916831	7·8780909	10·5196274	14·0274079	18·6791859	60
61	6·0683512	8·1538241	10·9404125	14·6586413	19·6131452	61
62	6·2504017	8·4392079	11·3780290	15·3182801	20·5938025	62
63	6·4379138	8·7345802	11·8331502	16·0076028	21·6234926	63
64	6·6310512	9·0402905	12·3064762	16·7279449	22·7046672	64
65	6·8299827	9·3567007	12·7987352	17·4807024	23·8399006	65
66	7·0348822	9·6841852	13·3106846	18·2673340	25·0318956	66
67	7·2459287	10·0231317	13·8431120	19·0893640	26·2834904	67
68	7·4633065	10·3739413	14·3968365	19·9483854	27·5976649	68
69	7·6872057	10·7370292	14·9727100	20·8460628	28·9775481	69
70	7·9178219	11·1128253	15·5716184	21·7841356	30·4264255	70
71	8·1553566	11·5017741	16·1944831	22·7644217	31·9477468	71
72	8·4000173	11 9043362	16·8422624	23·7888207	33·5451342	72
73	8·6520178	12·3209880	17·5159529	24·8593176	35·2223909	73
74	8·9115783	12·7522226	18·2165910	25·9779869	36·9835104	74
75	9·1789257	13·1985504	18·9452547	27·1469963	38·8326859	75
76	9·4542934	13·6604996	19·7030649	28·3686111	40·7743202	76
77	9·7379222	14·1386171	20·4911874	29·6451986	42·8130362	77
78	10·0300599	14·6334687	21·3108349	30·9792326	44·9536880	78
79	10·3309617	15·1456401	22·1632683	32·3732980	47·2013724	79
80	10·6408906	15·6757375	23·0497991	33·8300964	49·5614411	80
81	10·9601173	16·2243884	23·9717910	35·3524508	52·0395131	81
82	11·2889208	16·7922420	24·9306627	36·9433111	54·6414888	82
83	11·6275884	17·3799704	25·9278892	38·6057601	57·3735632	83
84	11·9764161	17·9882694	26·9650048	40·3430193	60·2422414	84
85	12·3357086	18·6178588	28·0436049	42·1584551	63·2543534	85
86	12·7057798	19·2694839	29·1653491	44·0555856	66·4170711	86
87	13·0869532	19·9439158	30·3319631	46·0380870	69·7379247	87
88	13·4795618	20·6419529	31·5452416	48·1098009	73·2248209	88
89	13·8839487	21·3644212	32·8070513	50·2747419	76·8860620	89
90	14·3004671	22·1121760	34·1193334	52·5371053	80·7303650	90
91	14·7294811	22·8861021	35·4841067	54·9012750	84·7668833	91
92	15·1713656	23·6871157	36·9034710	57·3718324	89·0052275	92
93	15·6265065	24·5161647	38·3796098	59·9535649	93·4554888	93
94	16·0953017	25·3742305	39·9147942	62·6514753	98·1282633	94
95	16·5781608	26·2623286	41·5113859	65·4707917	103·0346764	95
96	17·0755056	27·1815101	43·1718414	68·4169773	108·1864103	96
97	17·5877708	28·1328629	44·8987150	71·4957413	113·5957308	97
98	18·1154039	29·1175131	46·6946636	74·7130496	119·2755173	98
99	18·6588660	30·1366261	48·5624502	78·0751369	125·2392932	99
100	19·2186320	31·1914080	50·5049482	81·5885180	131·5012578	100
101	19·7951909	32·2831073	52·5251461	85·2600013	138·1763207	101
102	20·3890467	33·4130160	54·6261520	89·0967014	144·9801368	102
103	21·0007181	34·5824516	56·8111980	93·1060530	152·2291436	103
104	21·6307396	35·7928374	59·0836460	97·2958253	159·8406008	104

TABLE II. THE AMOUNT OF £1 PER ANNUM IN ANY NUMBER OF YEARS.

Years	3 per Cent.	3½ per Cent.	4 per Cent.	4½ per Cent.	5 per Cent	Years
1	1·000000	1·000000	1·000000	1·000000	1·000000	1
2	2·030000	2·035000	2·040000	2·045000	2·050000	2
3	3·090900	3·106225	3·121600	3·137025	3·152500	3
4	4·183627	4·214943	4·246464	4·278191	4·310125	4
5	5·309136	5·362466	5·416323	5·470710	5.525631	5
6	6·468410	6·550152	6·632975	6·716892	6.801913	6
7	7·662462	7·779408	7·898294	8·019152	8·142008	7
8	8·892336	9·051687	9·214226	9·380014	9·549109	8
9	10·159106	10·368496	10·582795	10·802114	11·026564	9
10	11·463879	11·731393	12·006107	12·288209	12·577893	10
11	12·807796	13·141992	13·486351	13·841179	14·206787	11
12	14·192030	14·601962	15·025805	15·464032	15·917127	12
13	15·617790	16·113030	16·626838	17·159913	17·712983	13
14	17·086324	17·676986	18·291911	18·932109	19·598632	14
15	18·598914	19·295681	20·023588	20·784054	21·578564	15
16	20·156881	20·971030	21·824531	22·719337	23·657492	16
17	21·761588	22·705016	23·697512	24·741707	25·840366	17
18	23·414435	24·499691	25·645413	26·855084	28·132385	18
19	25·116868	26·357181	27·671229	29·063562	30·539004	19
20	26·870374	28·279682	29·778079	31·371423	33·065954	20
21	28·676486	30·269471	31·969202	33·783137	35·719252	21
22	30·536780	32·328902	34·247970	36·303378	38·505214	22
23	32·452884	34·460414	36·617889	38·937030	41·430475	23
24	34·426470	36·666528	39·082604	41·689196	44·501999	24
25	36·459264	38·949857	41·645908	44·565210	47·727099	25
26	38·553042	41·313102	44·311745	47·570645	51·113454	26
27	40·709634	43·759060	47·084214	50·711324	54·669126	27
28	42·930923	46·290627	49·967583	53·993333	58·402583	28
29	45·218850	48·910799	52·966286	57·423033	62·322712	29
30	47·575416	51·622677	56·084938	61·007070	66·438848	30
31	50·002678	54·429471	59·328335	64·752388	70·760790	31
32	52·502759	57·334502	62·701469	68·666245	75.298829	32
33	55·077841	60·341210	66·209527	72·756226	80·063771	33
34	57·730177	63·453152	69·857909	77·030256	85·066959	34
35	60·462082	66·674013	73·652225	81·496618	90·320307	35
36	63·275944	70·007603	77·598314	86·163966	95·836323	36
37	66·174223	73·457869	81·702246	91·041344	101·628139	37
38	69·159449	77·028895	85·970336	96·138205	107·709546	38
39	72·234233	80·724906	90·409150	101·464424	114·095023	39
40	75·401260	84·550278	95·025516	107·030323	120·799774	40
41	78·663298	88·509537	99·826536	112·846688	127·839763	41
42	82·023196	92·607371	104·819598	118·924789	135·231751	42
43	85·483892	96·848629	110·012382	125·276404	142·993339	43
44	89·048409	101·238331	115·412877	131·913842	151·143006	44
45	92·719861	105·781673	121·029392	138·849965	159·700156	45
46	96·501457	110·484031	126·870568	146·098214	168·685164	46
47	100·396501	115·350973	132·945390	153·672633	178·119422	47
48	104·408396	120·388257	139·263206	161·587902	188·025393	48
49	108·540648	125·601846	145·833734	169·859357	198·426663	49
50	112·796867	130·997910	152·667084	178·503028	209·347996	50
51	117·180773	136·582837	159·773767	187·535665	220·815395	51
52	121·696197	142·363236	167·164718	196·974769	232·856165	52

TABLE II. THE AMOUNT OF £1 PER ANNUM IN ANY NUMBER OF YEARS.

Years	3 per Cent	3½ per Cent.	4 per Cent.	4½ per Cent.	5 per Cent.	Years
53	126·347082	148·345950	174·851306	206·838634	245·498974	53
54	131·137495	154·538058	182·845359	217·146373	258·773922	54
55	136·171620	160·946890	191·159173	227·917959	272·712618	55
56	141·153768	167·580001	199·805540	239·174268	287·348249	56
57	146·388381	174·445332	208·797762	250·937110	302·715662	57
58	151·780033	181·550919	218·149672	263·229280	318·851445	58
59	157·333434	188·905201	227·875659	276·074597	335·794017	59
60	163·053437	196·516883	237·990685	289·497954	353·583718	60
61	168·945040	204·394974	248·510313	303·525362	372·266904	61
62	175·013391	212·548798	259·450725	318·184003	391·879049	62
63	181·263793	220·988006	270·828754	333·502283	412·463851	63
64	187·701707	229·722586	282·661904	349·509886	434·093344	64
65	194·332758	238·762877	294·968381	366·237831	456·798011	65
66	201·162741	248·119577	307·767116	383·718533	480·637912	66
67	208·197623	257·803762	321·077800	401·985867	505·669807	67
68	215·443551	267·826894	334·920912	421·075231	531·953298	68
69	222·906858	278·200835	349·317749	441·023617	559·550963	69
70	230·594064	288·937865	364·290459	461·869680	588·528511	70
71	238·511886	300·050690	379·862077	483·653815	618·954936	71
72	246·667242	311·552464	396·056560	506·418237	650·902683	72
73	255·067259	323·456800	412·898823	530·207057	684·447817	73
74	263·719277	335·777788	430·414776	555·066375	719·670208	74
75	272·630856	348·530011	448·631367	581·044362	756·653718	75
76	281·809781	361·728561	467·576621	608·191358	795·486404	76
77	291·264075	375·389061	487·279686	636·559969	836·260725	77
78	301·001997	389·527678	507·770874	666·205168	879·073761	78
79	311·032057	404·161147	529·981708	697·184401	924·027449	79
80	321·363019	419·306787	551·244977	729·557699	971·228821	80
81	332·003909	434·982524	574·294776	763·387795	1020·790262	81
82	342·964026	451·206913	598·266567	798·740246	1072·829775	82
83	354·252947	467·999155	623·197230	835·683557	1127·471264	83
84	365·880536	485·379125	649·125119	874·289317	1184·844827	84
85	377·856952	503·367394	676·090124	914·632336	1245·087069	85
86	390·192660	521·985253	704·133728	956·790791	1308·341422	86
87	402·898440	541·254737	733·299078	1000·846377	1374·758493	87
88	415·985393	561·198653	763·631041	1046·884464	1444·496418	88
89	429·464955	581·840606	795·176282	1094·994265	1517·721239	89
90	443·348904	603·205027	827·983334	1145·269007	1594·607301	90
91	457·649371	625·317203	862·102667	1197·806112	1675·337666	91
92	472·378852	648·203305	897·586774	1252·707387	1760·104549	92
93	487·550217	671·890421	934·490245	1310·079219	1849·109777	93
94	503·176724	696·406585	972·869854	1370·032784	1942·565265	94
95	519·272026	721·780816	1012·784649	1432·684259	2040·693529	95
96	535·850186	748·043145	1054·296035	1498·155051	2143·728205	96
97	552·925692	775·224655	1097·467876	1566·572028	2251·914615	97
98	570·513463	803·357517	1142·366591	1638·067770	2365·510346	98
99	588·628867	832·475031	1189·061255	1712·780819	2484·785864	99
100	607·287733	862·611657	1237·623705	1790·855956	2610·025157	100
101	626·506365	893·813065	1288·128653	1872·444474	2741·526414	101
102	646·301556	926·096172	1340·653799	1957·704475	2879·702734	102
103	666·690603	959·509188	1395·279951	2046·801176	3024·682870	103
104	687·691321	994·091639	1452·091931	2139·907229	3176·912013	104

Years	3 per Cent.	3½ per Cent.	4 per Cent.	4½ per Cent.	5 per Cent.	Years
1	·9708738	·9661836	·9615385	·9569378	·9523810	1
2	·9425959	·9335107	·9245562	·9157300	·9070295	2
3	·9151417	·9019427	·8889964	·8762966	·8638376	3
4	·8884871	·8714422	·8548042	·8385613	·8227025	4
5	·8626088	·8419732	·8219271	·8024511	·7835262	5
6	·8374843	·8135006	·7903145	·7678957	·7462154	6
7	·8130915	·7859910	·7599178	·7348285	·7106813	7
8	·7894092	·7594116	·7306902	·7031851	·6768394	8
9	·7664167	·7337310	·7025867	·6729044	·6446089	9
10	·7440939	·7089188	·6755642	·6439277	·6139133	10
11	·7224213	·6849457	·6495809	·6161987	·5846793	11
12	·7013799	·6617833	·6245971	·5896639	·5568374	12
13	·6809513	·6394042	·6005741	·5642716	·5303214	13
14	·6611178	·6177818	·5774751	·5399729	·5050680	14
15	·6418620	·5968906	·5552645	·5167204	·4810171	15
16	·6231669	·5767059	·5339082	·4944693	·4581115	16
17	·6050165	·5572038	·5133733	·4731764	·4362967	17
18	·5873946	·5383611	·4936281	·4528004	·4155207	18
19	·5702860	·5201557	·4746424	·4333018	·3957340	19
20	·5536758	·5025659	·4563870	·4146429	·3768895	20
21	·5375493	·4855709	·4388336	·3967874	·3589424	21
22	·5218925	·4691506	·4219554	·3797009	·3418499	22
23	·5066918	·4532856	·4057263	·3633501	·3255713	23
24	·4919337	·4379571	·3901215	·3477035	·3100679	24
25	·4776056	·4231470	·3751168	·3327306	·2953028	25
26	·4636947	·4088377	·3606892	·3184025	·2812407	26
27	·4501891	·3950122	·3468166	·3046914	·2678483	27
28	·4370768	·3816543	·3334775	·2915707	·2550936	28
29	·4243464	·3687482	·3206514	·2790150	·2429463	29
30	·4119868	·3562784	·3083187	·2670000	·2313775	30
31	·3999871	·3442304	·2964603	·2555024	·2203595	31
32	·3883370	·3325897	·2850579	·2444999	·2098662	32
33	·3770263	·3213427	·2740942	·2339712	·1998725	33
34	·3660449	·3104761	·2635521	·2238959	·1903548	34
35	·3553834	·2999769	·2534155	·2142544	·1812903	35
36	·3450324	·2898327	·2436687	·2050282	·1726574	36
37	·3349829	·2800316	·2342969	·1961992	·1644356	37
38	·3252261	·2705619	·2252854	·1877504	·1566054	38
39	·3157536	·2614125	·2166206	·1796655	·1491480	39
40	·3065568	·2525725	·2082890	·1719287	·1420457	40
41	·2976280	·2440314	·2002779	·1645251	·1352816	41
42	·2889592	·2357791	·1925749	·1574403	·1288396	42
43	·2805429	·2278059	·1851682	·1506605	·1227044	43
44	·2723718	·2201023	·1780464	·1441728	·1168613	44
45	·2644386	·2126592	·1711984	·1379644	·1112965	45
46	·2567365	·2054679	·1646139	·1320233	·1059967	46
47	·2492588	·1985197	·1582826	·1263381	·1009492	47
48	·2419988	·1918065	·1521948	·1208977	·0961421	48
49	·2349503	·1853202	·1463411	·1156916	·0915639	49
50	·2281071	·1790534	·1407126	·1107097	·0872037	50
51	·2214632	·1729984	·1353006	·1059423	·0830512	51
52	·2150128	·1671482	·1300967	·1013801	·0790964	52

Years	3 per Cent.	3½ per Cent.	4 per Cent.	4½ per Cent.	5 per Cent.	Years
53	·2087503	·1614959	·1250930	·0970145	·0753299	53
54	·2026702	·1560347	·1202817	·0928368	·0717427	54
55	·1967672	·1507581	·1156555	·0888391	·0683264	55
56	·1910361	·1456600	·1112072	·0850135	·0650728	56
57	·1854719	·1407343	·1069300	·0813526	·0619740	57
58	·1800698	·1359752	·1028173	·0778494	·0590229	58
59	·1748251	·1313770	·0988628	·0744970	·0562123	59
60	·1697331	·1269343	·0950604	·0712890	·0535355	60
61	·1647894	·1226418	·0914042	·0682192	·0509862	61
62	·1599897	·1184945	·0878887	·0652815	·0485583	62
63	·1553298	·1144875	·0845084	·0624703	·0462460	63
64	·1508057	·1106159	·0812580	·0597802	·0440438	64
65	·1464133	·1068753	·0781327	·0572059	·0419465	65
66	·1421488	·1032611	·0751276	·0547425	·0399490	66
67	·1380085	·0997692	·0722381	·0523852	·0380467	67
68	·1339889	·0963954	·0694597	·0501294	·0362350	68
69	·1300863	·0931356	·0667882	·0479707	·0345095	69
70	·1262974	·0899861	·0642194	·0459050	·0328662	70
71	·1226188	·0869431	·0617494	·0439282	·0313011	71
72	·1190474	·0840030	·0593745	·0420366	·0298106	72
73	·1155800	·0811623	·0570908	·0402264	·0283910	73
74	·1122136	·0784177	·0548950	·0384941	·0270391	74
75	·1089452	·0757659	·0527837	·0368365	·0257515	75
76	·1057721	·0732038	·0507535	·0352502	·0245252	76
77	·1026913	·0707283	·0488015	·0337323	·0233574	77
78	·0997003	·0683365	·0469245	·0322797	·0222451	78
79	·0967964	·0660256	·0451197	·0308897	·0211850	79
80	·0939771	·0637929	·0433843	·0295595	·0201770	80
81	·0912399	·0616356	·0417157	·0282866	·0192162	81
82	·0885824	·0595513	·0401113	·0270685	·0183011	82
83	·0860024	·0575375	·0385685	·0259029	·0174296	83
84	·0834974	·0555918	·0370851	·0247874	·0165997	84
85	·0810655	·0537119	·0356588	·0237200	·0158092	85
86	·0787043	·0518955	·0342877	·0226986	·0150564	86
87	·0764120	·0501406	·0329685	·0217212	·0143394	87
88	·0741864	·0484450	·0317005	·0207858	·0136566	88
89	·0720256	·0468068	·0304813	·0198907	·0130063	89
90	·0699278	·0452240	·0293089	·0190342	·0123869	90
91	·0678911	·0436946	·0281816	·0182145	·0117971	91
92	·0659136	·0422170	·0270977	·0174302	·0112303	92
93	·0639938	·0407894	·0260555	·0166796	·0107003	93
94	·0621299	·0394101	·0250534	·0159613	·0101907	94
95	·0603203	·0380774	·0240898	·0152740	·0097055	95
96	·0585634	·0367897	·0231633	·0146163	·0092433	96
97	·0568577	·0355456	·0222724	·0139869	·0088032	97
98	·0552016	·0343436	·0214157	·0133845	·0083840	98
99	·0535938	·0331822	·0205920	·0128082	·0079847	99
100	·0520328	·0320601	·0198000	·0122566	·0076045	100
101	·0505173	·0309760	·0190385	·0117288	·0072424	101
102	·0490459	·0299285	·0183063	·0112238	·0068975	102
103	·0476174	·0289164	·0176022	·0107404	·0065690	103
104	·0462305	·0279385	·0169252	·0102779	·0062561	104

TABLE IV. THE PRESENT VALUE OF £1 PER ANNUM FOR ANY NUMBER OF YEARS.

Years	3 per Cent.	3½ per Cent.	4 per Cent.	4½ per Cent.	5 per Cent.	Years
1	·970874	·966184	·961538	·956938	·952381	1
2	1·913470	1·899694	1·886095	1·872668	1·859410	2
3	2 828611	2·801637	2·775091	2·748964	2·723248	3
4	3·717098	3·673079	3·629895	3·587526	3·545951	4
5	4 579707	4·515052	4·451822	4·389977	4·329477	5
6	5·417191	5·328553	5·242137	5·157872	5·075692	6
7	6·230283	6·114544	6·002055	5·892701	5·786373	7
8	7 019692	6·873956	6·732745	6·595886	6·463213	8
9	7·786109	7·607687	7·435332	7·268790	7·107822	9
10	8·530203	8·316605	8·110896	7·912718	7·721735	10
11	9·252624	9·001551	8·760477	8·528917	8·306414	11
12	9·954004	9·663334	9·385074	9·118581	8·863252	12
13	10·634955	10·302738	9·985648	9·682852	9·393573	13
14	11·296073	10·920520	10·563123	10·222825	9·898641	14
15	11·937935	11·517411	11·118387	10·739546	10·379658	15
16	12·561102	12·094117	11·652296	11·234015	10·837770	16
17	13·166118	12·651321	12·165669	11·707191	11·274066	17
18	13·753513	13·189682	12·659297	12·159992	11·689587	18
19	14·323799	13·709837	13·133939	12·593294	12·085321	19
20	14·877475	14·212403	13·590326	13·007936	12·462210	20
21	15·415024	14·697974	14·029160	13·404724	12·821153	21
22	15·936917	15·167125	14·451115	13·784425	13·163003	22
23	16·443608	15·620410	14·856842	14·147775	13·488574	23
24	16·935542	16·058368	15·246963	14·495478	13·798642	24
25	17·413148	16·481515	15·622080	14·828209	14·093945	25
26	17·876842	16·890352	15·982769	15·146611	14·375185	26
27	18·327031	17·285365	16·329586	15·451303	14·643034	27
28	18·764108	17·667019	16·663063	15·742874	14·898127	28
29	19·188455	18·035767	16·983715	16·021889	15·141074	29
30	19·600441	18·392045	17·292033	16·288889	15·372451	30
31	20·000428	18·736276	17·588494	16·544391	15·592811	31
32	20·388766	19·068865	17·873552	16·788891	15·802677	32
33	20·765792	19·390208	18·147646	17.022862	16·002549	33
34	21·131837	19·700684	18·411198	17.246758	16·192904	34
35	21·487220	20·000661	18·664613	17·461012	16·374194	35
36	21·832252	20·290494	18·908282	17·666041	16·546852	36
37	22·167235	20·570525	19·142579	17·822240	16·711287	37
38	22·492462	20·841087	19·367864	18·049990	16·867893	38
39	22·808215	21·102500	19·584485	18·229656	17·017041	39
40	23·114772	21·355072	19·792774	18·401584	17·159086	40
41	23·412400	21·599104	19·993052	18·566109	17·294368	41
42	23·701359	21·834883	20·185627	18·723550	17·423208	42
43	23·901902	22·062689	20·370795	18·874210	17·545912	43
44	24·254274	22·282791	20·548841	19·018383	17·662773	44
45	24·518713	22·495450	20·720040	19·156347	17·774070	45
46	24·775449	22·700918	20·884654	19·288371	17·880067	46
47	25·024708	22·899438	21·042936	19·414709	17·981016	47
48	25·266707	23·091244	21·195131	19·535607	18·077158	48
49	25·501657	23·276564	21·341472	19·651298	18·168722	49
50	25·729764	23·455618	21·482185	19·762008	18·255925	50
51	25·951227	23·628616	21·617485	19.867950	18·338977	51
52	26·166240	23·795765	21·747582	19·969330	18·418073	52

THE PRESENT VALUE OF £1 PER ANNUM FOR ANY NUMBER OF YEARS.

Years	3 per Cent.	3½ per Cent.	4 per Cent.	4½ per Cent.	5 per Cent.	Years
53	26·374990	23·957260	21·872675	20·066345	18·493403	53
54	26·577660	24·113295	21·992957	20·159181	18·565146	54
55	26·774428	24·264053	22·108612	20·248021	18·633472	55
56	26·965464	24·409713	22·219819	20·333034	18·698545	56
57	27·150936	24·550448	22·326749	20·414387	18·760519	57
58	27·331005	24·686423	22·429567	20·492236	18 819542	58
59	27·505831	24·817800	22·528430	20·566733	18·875754	59
60	27·675564	24·944734	22·623490	20·638022	18·929290	60
61	27·840353	25·067376	22·714894	20·706241	18·980276	61
62	28·000343	25·185870	22·802783	20·771523	19·028834	62
63	28·155673	25·300358	22·887291	20·833993	19·075080	63
64	28·306478	25·410974	22·968549	20·893773	19·119124	64
65	28·452891	25·517849	23·046682	20·950979	19·161070	65
66	28·595040	25·621110	23·121810	21·005722	19·201019	66
67	28·733049	25·720880	23·194048	21·058107	19·239066	67
68	28·867038	25·817275	23·263507	21·108236	19·275301	68
69	28·997124	25·910411	23·330296	21·156207	19·309810	69
70	29·123421	26·000397	23·394515	21·202112	19·342677	70
71	29·246040	26·087340	23·456264	21·246040	19·373978	71
72	29·365087	26·171343	23·515639	21·288077	19·403788	72
73	29·480667	26·252505	23·572730	21·328303	19·432179	73
74	29.592881	26·330923	23·627625	21·366797	19·459218	74
75	29·701826	26·406689	23·680408	21·403634	19·484970	75
76	29·807598	26·479892	23·731162	21·438884	19·509495	76
77	29 910290	26·550621	23·779963	21·472616	19·532853	77
78	30·009990	26·618957	23·826888	21·504896	19·555098	78
79	30·106786	26·684983	23·872008	21·535785	19·576284	79
80	30·200763	26·748776	23·915392	21·565345	19·596460	80
81	30·292003	26·810411	23·957108	21·593632	19·615677	81
82	30·380586	26·869963	23·997219	21·620700	19·633978	82
83	30·466588	26·927500	24·035787	21·646603	19·651407	83
84	30·550086	26·983092	24·072872	21·671390	19·668007	84
85	30·631151	27·036804	24·108531	21·695110	19·683816	85
86	30·709855	27·088699	24·142818	21·717809	19·698873	86
87	30·786267	27·138840	24.175787	21·739530	19·713212	87
88	30·860454	27·187285	24·207487	21·760316	19·726869	88
89	30·932479	27·234092	24·237969	21·780207	19·739875	89
90	31·002407	27·279316	24·267278	21·799241	19·752262	90
91	31·070298	27·323010	24·295459	21·817455	19·764059	91
92	31·136212	27·365227	24·322557	21·834885	19·775294	92
93	31·200206	27·406017	24·348612	21·851565	19·785994	93
94	31·262336	27·445427	24·373666	21·867526	19·796185	94
95	31·322656	27·483504	24·397756	21·882800	19·805891	95
96	31·381219	27·520294	24·420919	21·897417	19·815134	96
97	31·438077	27·550839	24·443191	21·911403	19·823937	97
98	31·493279	27·590183	24·464607	21·924788	19·832321	98
99	31·546872	27·623365	24·485199	21·937596	19·840306	99
100	31·598905	27·655425	24·504999	21·949853	19·847910	100
101	31·649422	27·686401	24·524038	21·961582	19·855152	101
102	31·698468	27·716329	24·542344	21·972806	19·862050	102
103	31·746086	27·745246	24·559946	21·983546	19·868619	103
104	31·792316	27·773184	24·576871	21·993824	19·874875	104

TABLE V. THE ANNUITY WHICH £1 WILL PURCHASE FOR ANY NUMBER OF YEARS.

Years	3 per Cent.	3½ per Cent.	4 per Cent.	4½ per Cent.	5 per Cent.	Years
1	1·0300000	1·0350000	1·0400000	1·0450000	1·0500000	1
2	0·5226108	0·5264005	0·5301961	0·5339976	0·5378049	2
3	·3535304	·3569342	·3603485	·3637734	·3672086	3
4	·2690271	·2722511	·2754901	·2787437	·2820118	4
5	·2183546	·2214814	·2246271	·2277916	·2309748	5
6	·1845975	·1876682	·1907619	·1938784	·1970175	6
7	·1605064	·1635445	·1666096	·1697015	·1728198	7
8	·1424564	·1454767	·1485278	·1516097	·1547218	8
9	·1284339	·1314460	·1344930	·1375745	·1406901	9
10	·1172305	·1202414	·1232909	·1263788	·1295046	10
11	·1080775	·1110920	·1141490	·1172482	·1203889	11
12	·1004621	·1034840	·1065522	·1096662	·1128254	12
13	·0940295	·0970616	·1001437	·1032754	·1064558	13
14	·0885263	·0915707	·0946690	·0978203	·1010240	14
15	·0837666	·0868251	·0899411	·0931138	·0963423	15
16	·0796109	·0826848	·0858200	·0890154	·0922699	16
17	·0759525	·0790431	·0821985	·0854176	·0886991	17
18	·0727087	·0758168	·0789933	·0822369	·0855462	18
19	·0698139	·0729403	·0761386	·0794073	·0827450	19
20	·0672157	·0703611	·0735818	·0768761	·0802426	20
21	·0648718	·0680366	·0712801	·0746006	·0779961	21
22	·0627474	·0659321	·0691988	·0725457	·0757075	22
23	·0608139	·0640188	·0673091	·0706825	·0741368	23
24	·0590474	·0622728	·0655868	·0689870	·0724709	24
25	·0574279	·0606740	·0640120	·0674390	·0709525	25
26	·0559383	·0592054	·0625674	·0660214	·0695643	26
27	·0545642	·0578524	·0612385	·0647195	·0682919	27
28	·0532932	·0566027	·0600130	·0635208	·0671225	28
29	·0521147	·0554454	·0588799	·0624146	·0660455	29
30	·0510193	·0543713	·0578301	·0613915	·0650514	30
31	·0499989	·0533724	·0568554	·0604435	·0641321	31
32	·0490466	·0524415	·0559486	·0595632	·0632804	32
33	·0481561	·0515724	·0551036	·0587445	·0624900	33
34	·0473220	·0507597	·0543148	·0579819	·0617555	34
35	·0465393	·0499983	·0535773	·0572705	·0610717	35
36	·0458038	·0492842	·0528869	·0566058	·0604345	36
37	·0451116	·0486133	·0522396	·0559840	·0598398	37
38	·0444593	·0479821	·0516319	·0554017	·0592842	38
39	·0438439	·0473878	·0510608	·0548557	·0587646	39
40	·0432624	·0468273	·0505235	·0543432	·0582782	40
41	·0427124	·0462982	·0500174	·0538616	·0578223	41
42	·0421917	·0457983	·0495402	·0534087	·0573947	42
43	·0416981	·0453254	·0490899	·0529824	·0569933	43
44	·0412299	·0448777	·0486645	·0525807	·0566163	44
45	·0407852	·0444534	·0482625	·0522020	·0562617	45
46	·0403625	·0440511	·0478821	·0518447	·0559282	46
47	·0399605	·0436692	·0475219	·0515073	·0556142	47
48	·0395778	·0433065	·0471807	·0511886	·0553184	48
49	·0392131	·0429617	·0468571	·0508872	·0550397	49
50	·0388655	·0426337	·0465502	·0506022	·0547767	50
51	·0385338	·0423216	·0462589	·0503323	·0545287	51
52	·0382172	·0420243	·0459821	·0500768	·0542945	52

TABLE V. THE ANNUITY WHICH £1 WILL PURCHASE FOR ANY NUMBER OF YEARS.

Years	3 per Cent.	3½ per Cent.	4 per Cent.	4½ per Cent.	5 per Cent.	Years
53	0·0379147	0·0417410	0·0457192	0·0498347	0·0540733	53
54	·0376256	·0414709	·0454691	·0496052	·0538644	54
55	·0373491	·0412132	·0452312	·0493875	·0536669	55
56	·0370845	·0409673	·0450049	·0491811	·0534801	56
57	·0368311	·0407325	·0447893	·0489851	·0533034	57
58	·0365885	·0405081	·0445840	·0487990	·0531363	58
59	·0363559	·0402937	·0443884	·0486222	·0529780	59
60	·0361330	·0400886	·0442019	·0484543	·0528282	60
61	·0359191	·0398925	·0440240	·0482946	·0526863	61
62	·0357139	·0397048	·0438543	·0481428	·0525518	62
63	·0355168	·0395251	·0436924	·0479985	·0524244	63
64	·0353276	·0393531	·0435378	·0478612	·0523037	64
65	·0351458	·0391883	·0433902	·0477305	·0521892	65
66	·0349711	·0390303	·0432492	·0476061	·0520806	66
67	·0348031	·0388789	·0431145	·0474877	·0519776	67
68	·0346416	·0387338	·0429858	·0473749	·0518799	68
69	·0344862	·0385945	·0428627	·0472675	·0517872	69
70	·0343366	·0384610	·0427451	·0471651	·0516992	70
71	·0341927	·0383328	·0426325	·0470676	·0516156	71
72	·0340540	·0382097	·0425249	·0469747	·0515363	72
73	·0339205	·0380916	·0424219	·0468861	·0514610	73
74	·0337919	·0379782	·0423233	·0468016	·0513895	74
75	·0336680	·0378692	·0422290	·0467210	·0513216	75
76	·0335485	·0377645	·0421387	·0466442	·0512571	76
77	·0334333	·0376639	·0420522	·0465709	·0511958	77
78	·0333222	·0375672	·0419694	·0465010	·0511376	78
79	·0332151	·0374743	·0418901	·0464343	·0510822	79
80	·0331118	·0373849	·0418141	·0463707	·0510296	80
81	·0330120	·0372989	·0417413	·0463100	·0509796	81
82	·0329158	·0372163	·0416715	·0462520	·0509321	82
83	·0328228	·0371368	·0416046	·0461966	·0508869	83
84	·0327331	·0370603	·0415405	·0461438	·0508440	84
85	·0326465	·0369866	·0414791	·0460933	·0508032	85
86	·0325628	·0369158	·0414202	·0460452	·0507643	86
87	·0324820	·0368476	·0413637	·0459992	·0507274	87
88	·0324039	·0367819	·0413095	·0459552	·0506923	88
89	·0323285	·0367187	·0412576	·0459133	·0506589	89
90	·0322556	·0366578	·0412078	·0458732	·0506271	90
91	·0321851	·0365992	·0411600	·0458349	·0505969	91
92	·0321170	·0365427	·0411141	·0457983	·0505682	92
93	·0320511	·0364883	·0410701	·0457633	·0505408	93
94	·0319874	·0364359	·0410279	·0457299	·0505148	94
95	·0319258	·0363855	·0409874	·0456980	·0504900	95
96	·0318662	·0363368	·0409485	·0456675	·0504665	96
97	·0318086	·0362900	·0409112	·0456383	·0504441	97
98	·0317528	·0362448	·0408754	·0456105	·0504227	98
99	·0316989	·0362012	·0408410	·0455839	·0504025	99
100	·0316467	·0361593	·0408080	·0455584	·0503831	100
101	·0315961	·0361188	·0407763	·0455341	·0503648	101
102	·0315473	·0360798	·0407457	·0455108	·0503473	102
103	·0314999	·0360422	·0407167	·0454886	·0503306	103
104	·0314541	·0360060	·0406887	·0454673	·0503148	104

Table VI. LOGARITHM OF THE PRESENT VALUE OF £1, DUE ANY NUMBER OF YEARS.

Years	3 per Cent.	3½ per Cent.	4 per Cent.	4½ per Cent.	5 per Cent.	Years
1	1·9871628	1·9850597	1·9829667	1·9808837	1·9788107	1
2	1·9743256	1·9701193	1·9659333	1·9617674	1·9576214	2
3	1·9614883	1·9551790	1·9489000	1·9426511	1·9364321	3
4	1·9486511	1·9402386	1·9318666	1·9235348	1·9152428	4
5	1·9358139	1·9252983	1·9148333	1·9044185	1·8940535	5
6	1·9229767	1·9103579	1·8978000	1·8853023	1·8728642	6
7	1·9101394	1·8954176	1·8807666	1·8661860	1·8516749	7
8	1·8973022	1·8804772	1·8637333	1·8470697	1·8304856	8
9	1·8844650	1·8655369	1·8466999	1·8279534	1·8092963	9
10	1·8716278	1·8505965	1·8296666	1·8088371	1·7881070	10
11	1·8587906	1·8356562	1·8126333	1·7897208	1·7669177	11
12	1·8459533	1·8207158	1·7955999	1·7706045	1·7457284	12
13	1·8331161	1·8057755	1·7785666	1·7514882	1·7245391	13
14	1·8202789	1·7908351	1·7615332	1·7323719	1·7033498	14
15	1·8074416	1·7758948	1·7444999	1·7132556	1·6821605	15
16	1·7946044	1·7609544	1·7274666	1·6941394	1·6609712	16
17	1·7817672	1·7460141	1·7104332	1·6750231	1·6397819	17
18	1·7689300	1·7310737	1·6933999	1·6559068	1·6185926	18
19	1·7560927	1·7161334	1·6763666	1·6367905	1·5974033	19
20	1·7432555	1·7011930	1·6593332	1·6176742	1·5762140	20
21	1·7304183	1·6862527	1·6422999	1·5985579	1·5550247	21
22	1·7175811	1·6713123	1·6252665	1·5794416	1·5338354	22
23	1·7047438	1·6563720	1·6082332	1·5603253	1·5126461	23
24	1·6919066	1·6414316	1·5911999	1·5412090	1·4914568	24
25	1·6790694	1·6264913	1·5741665	1·5220927	1·4702675	25
26	1·6662322	1·6115509	1·5571332	1·5029764	1·4490782	26
27	1·6533949	1·5966106	1·5400998	1·4838602	1·4278889	27
28	1·6405577	1·5816702	1·5230665	1·4647439	1·4066996	28
29	1·6277205	1·5667299	1·5060332	1·4456276	1·3855103	29
30	1·6148833	1·5517895	1·4889998	1·4265113	1·3643210	30
31	1·6020460	1·5368492	1·4719665	1·4073950	1·3431317	31
32	1·5892088	1·5219088	1·4549331	1·3882787	1·3219424	32
33	1·5763716	1·5069685	1·4378998	1·3691624	1·3007531	33
34	1·5635344	1.4920281	1·4208665	1·3500461	1·2795638	34
35	1·5506971	1·4770878	1·4038331	1·3309298	1·2583745	35
36	1·5378599	1·4621474	1·3867998	1·3118135	1·2371852	36
37	1·5250227	1·4472071	1·3697664	1·2926973	1·2159959	37
38	1·5121855	1·4322667	1·3527331	1·2735810	1·1948066	38
39	1·4993482	1·4173264	1·3356998	1·2544647	1·1736173	39
40	1·4865110	1·4023860	1·3186664	1·2353484	1·1524280	40
41	1·4736738	1·3874457	1·3016331	1·2162321	1·1312387	41
42	1·4608366	1·3725053	1·2845997	1·1971158	1·1100494	42
43	1·4479993	1·3575650	1·2675664	1·1779995	1·0888601	43
44	1·4351621	1·3426246	1·2505331	1·1588832	1·0676708	44
45	1·4223249	1·3276843	1·2334997	1·1397669	1·0464815	45
46	1·4094877	1·3127439	1·2164664	1·1206506	1·0252922	46
47	1·3966504	1·2978036	1·1994331	1·1015343	1·0041029	47
48	1·3838132	1·2828632	1·1823997	1·0824181	2·9829136	48
49	1·3709760	1·2679229	1·1653664	1·0633018	2·9617243	49
50	1·3581388	1·2529825	1·1483330	1·0441855	2·9405350	50
51	1·3453015	1·2380422	1·1312997	1·0250692	2·9193457	51
52	1·3324643	1·2231018	1·1142664	1·0059529	2·8981564	52

TABLE VI. LOGARITHM OF THE PRESENT VALUE OF £1, DUE ANY NUMBER OF YEARS.

Years	3 per Cent.	3½ per Cent.	4 per Cent.	4½ per Cent.	5 per Cent.	Years
53	1·3196271	1·2081615	1·0972330	2·9868366	2·8769671	53
54	1·3067899	1·1932211	1·0801997	2·9677203	2·8557779	54
55	1·2939526	1·1782808	1·0631663	2·9486040	2·8345886	55
56	1·2811154	1·1633404	1·0461330	2·9294877	2·8133993	56
57	1·2682782	1·1484001	1·0290997	2·9103714	2·7922100	57
58	1·2554410	1·1334597	1·0120663	2·8912552	2·7710207	58
59	1·2426037	1·1185194	1·9950330	2·8721389	2·7498314	59
60	1·2297665	1·1035790	1·9779996	2·8530226	2·7286421	60
61	1·2169293	1·0886387	1·9609663	2·8339063	2·7074528	61
62	1·2040921	1·0736983	1·9439330	2·8147900	2·6862635	62
63	1·1912548	1·0587580	1·9268996	2·7956737	2·6650742	63
64	1·1784176	1·0438176	1·9098663	2·7765574	2·6438849	64
65	1·1655804	1·0288773	1·8928329	2·7574411	2·6226956	65
66	1·1527432	1·0139369	1·8757996	6·7383248	2·6015063	66
67	1·1399059	2·9989966	1·8587663	2·7192085	2·5803170	67
68	1·1270687	2·9840562	1·8417329	2·7000922	2·5591277	68
69	1·1142315	2·9691159	1·8246996	2·6809760	2·5379384	69
70	1·1013943	2·9511755	1·8076662	2·6618597	2·5167491	70
71	1·0885570	2·9392352	1·7906329	2·6427434	2·4955598	71
72	1·0757198	2·9242948	1·7735996	2·6236271	2·4743705	72
73	1·0628826	2·9093545	1·7565662	2·6045108	2·4531812	73
74	1·0500454	2·8944141	1·7395329	2·5853945	2·4319919	74
75	1·0372081	2·8794738	1·7224996	2·5662782	2·4108026	75
76	1·0243709	2·8645334	1·7054662	2·5471619	2·3896133	76
77	1·0115337	2·8495931	1·6884329	2·5280456	2·3684240	77
78	1·9986965	2·8346527	1·6713995	2·5089293	2·3472347	78
79	1·9858592	2·8197124	1·6543662	2·4898131	2·3260454	79
80	1·9730220	2·8047720	1·6373329	2·4706968	2·3048561	80
81	1·9601848	2·7898317	1·6202995	2·4515805	2·2836668	81
82	1·9473476	2·7748913	1·6032662	2·4324642	2·2624775	82
83	1·9345103	2·7599510	1·5862328	2·4133479	2·2412882	83
84	1·9216731	2·7450106	1·5691995	2·3942316	2·2200989	84
85	1·9088359	2·7300703	1·5521662	2·3751153	2·1989096	85
86	1·8959987	2·7151299	1·5351328	2·3559990	2·1777203	86
87	1·8831615	2·7001896	1·5180995	2·3368827	2·1565310	87
88	1·8703242	2·6852492	1·5010661	2·3177664	2·1353417	88
89	1·8574870	2·6703089	1·4840328	2·2986502	2·1141524	89
90	1·8446498	2·6553685	1·4669995	2·2795339	2·0929631	90
91	1·8318126	2·6404282	1·4499661	2·2604176	2·0717738	91
92	1·8189753	2·6254878	1·4329328	2·2413013	2·0505845	92
93	1·8061381	2·6105475	1·4158994	2·2221850	2·0293952	93
94	1·7933009	2·5956071	1·3988661	2·2030687	2·0082059	94
95	1·7804637	2·5806668	1·3818328	2·1839524	3·9870166	95
96	1·7676264	2·5657264	1·3647994	2·1648361	3·9658273	96
97	1·7547892	2·5507861	1·3477661	2·1457198	3·9446380	97
98	1·7419520	2·5358457	1·3307327	2·1266035	3·9234487	98
99	1·7291148	2·5209054	1·3136994	2·1074872	3·9022594	99
100	1·7162775	2·5059650	1·2966661	2·0883710	3·8810701	100
101	1·7034403	2·4910247	1·2796327	2·0692547	3·8598808	101
102	1·6906031	2·4760843	1·2625994	2·0501384	3·8386915	102
103	1·6777659	2·4611440	1·2455661	2·0310221	3·8175022	103
104	1·6649286	2·4462036	1·2285327	2·0119058	3·7963129	104

NORTHAMPTON, THE EQUITABLE, AND DES PARCIEUX.

Age.	Northampton.		Des Parcieux.		Equitable.	
	l_x	d_x	l_x	d_x	l_x	d_x
0	11650	3000				
1	8650	1367				
2	7283	502				
3	6781	335	1000	30		
4	6446	197	970	22		
5	6249	184	948	18		
6	6065	140	930	15		
7	5925	110	915	13		
8	5815	80	902	12		
9	5735	60	890	10		
10	5675	52	880	8	2844	11
11	5623	50	872	6	2833	11
12	5573	50	866	6	2822	12
13	5523	50	860	6	2810	12
14	5473	50	854	6	2798	13
15	5423	50	848	6	2785	14
16	5373	53	842	7	2771	15
17	5320	58	835	7	2756	16
18	5262	63	828	7	2740	17
19	5199	67	821	7	2723	18
20	5132	72	814	8	2705	18
21	5060	75	806	8	2687	18
22	4985	75	798	8	2669	19
23	4910	75	790	8	2650	19
24	4835	75	782	8	2631	20
25	4760	75	774	8	2611	20
26	4685	75	766	8	2591	21
27	4610	75	758	8	2570	22
28	4535	75	750	8	2548	23
29	4460	75	742	8	2525	24
30	4385	75	734	8	2501	24
31	4310	75	726	8	2477	25
32	4235	75	718	8	2452	26
33	4160	75	710	8	2426	26
34	4085	75	702	8	2400	26
35	4010	75	694	8	2374	27
36	3935	75	686	8	2347	27
37	3860	75	678	7	2320	28
38	3785	75	671	7	2292	28
39	3710	75	664	7	2264	28
40	3635	76	657	7	2236	28
41	3559	77	650	7	2208	28
42	3482	78	643	7	2180	28
43	3404	78	636	7	2152	29
44	3326	78	629	7	2123	30
45	3248	78	622	7	2093	30
46	3170	78	615	8	2063	30
47	3092	78	607	8	2033	31
48	3014	78	599	9	2002	32
49	2936	79	590	9	1970	33
50	2857	81	581	10	1937	35
51	2776	82	571	11	1902	37
52	2694	82	560	11	1865	39
53	2612	82	549	11	1826	41
54	2530	82	538	12	1785	41
55	2448	82	526	12	1744	42
56	2366	82	514	12	1702	43
57	2284	82	502	13	1659	44
58	2202	82	489	13	1615	45
59	2120	82	476	13	1570	46
60	2038	82	463	13	1524	46
61	1956	82	450	13	1478	46
62	1874	81	437	14	1432	47
63	1793	81	423	14	1385	48
64	1712	80	409	14	1337	49
65	1632	80	395	15	1288	50
66	1552	80	380	16	1238	51
67	1472	80	364	17	1187	52
68	1392	80	347	18	1135	53
69	1312	80	329	19	1082	54
70	1232	80	310	19	1028	54
71	1152	80	291	20	974	55
72	1072	80	271	20	919	55
73	992	80	251	20	864	56
74	912	80	231	20	808	56
75	832	80	211	19	752	55
76	752	77	192	19	697	55
77	675	73	173	19	642	54
78	602	68	154	18	588	54
79	534	65	136	18	534	54
80	469	63	118	17	480	54
81	406	60	101	16	426	53
82	346	57	85	14	373	52
83	289	55	71	12	321	50
84	234	48	59	11	271	47
85	186	41	48	10	224	43
86	145	34	38	9	181	38
87	111	28	29	7	143	32
88	83	21	22	6	111	26
89	62	16	16	5	85	20
90	46	12	11	4	65	16
91	34	10	7	3	49	13
92	24	8	4	2	36	11
93	16	7	2	1	25	9
94	9	5	1	1	16	7
95	4	3	0	0	9	5
96	1	1			4	3
97					1	1

TABLE VIII. RATE OF MORTALITY IN SWEDEN, BOTH MALES AND FEMALES.

Age.	Males.		Females.		Total.		Age.	Males.		Females.		Total.	
	l_x	d_x	l_x	d_x	l_x	d_x		l_x	d_x	l_x	d_x	l_x	d_x
0	10000	2300	10000	2090	10000	2195	49	3751	85	4097	70	3924	78
1	7700	500	7910	518	7805	509	50	3666	95	4027	75	3846	85
2	7200	337	7392	350	7296	344	51	3571	95	3952	80	3761	87
3	6863	240	7042	250	6952	245	52	3476	95	3872	85	5674	90
4	6623	150	6792	135	6707	143	53	3381	95	3787	85	3584	90
5	6473	125	6657	120	6564	122	54	3286	95	3702	85	3494	91
6	6348	105	6537	104	6442	105	55	3191	95	3617	85	3403	91
7	6243	90	6432	85	6337	87	56	3096	95	3532	85	3312	92
8	6153	75	6347	70	6250	73	57	3001	100	3447	90	3220	95
9	6078	65	6277	60	6177	62	58	2901	100	3357	90	3125	95
10	6013	55	6217	52	6115	54	59	2801	100	3267	100	3030	100
11	5958	45	6165	46	6061	45	60	2701	105	3167	110	2930	108
12	5913	45	6119	40	6016	42	61	2596	110	3057	118	2822	114
13	5868	40	6079	35	5974	38	62	2486	115	2939	120	2708	118
14	5828	40	6044	35	5936	37	63	2371	115	2819	120	2590	118
15	5788	39	6009	35	5899	37	64	2256	115	2699	120	2472	118
16	5749	39	5974	40	5862	40	65	2141	115	2579	120	2354	118
17	5710	39	5934	40	5822	40	66	2026	115	2458	120	2236	118
18	5671	44	5894	42	5782	42	67	1911	120	2339	120	2118	121
19	5627	44	5852	43	5742	43	68	1791	125	2219	120	1997	124
20	5583	50	5809	43	5697	47	69	1666	125	2099	120	1873	124
21	5533	50	5766	43	5650	47	70	1541	125	1979	130	1749	127
22	5483	50	5723	43	5603	48	71	1416	125	1849	140	1622	133
23	5433	55	5680	44	5555	48	72	1291	120	1709	150	1489	135
24	5378	55	5636	45	5507	50	73	1171	120	1559	160	1354	140
25	5323	55	5591	45	5457	50	74	1051	110	1399	150	1214	130
26	5268	55	5546	50	5407	52	75	941	105	1249	140	1084	121
27	5213	55	5499	52	5355	54	76	836	100	1109	130	963	115
28	5158	55	5444	55	5301	55	77	736	90	979	120	848	105
29	5103	56	5389	55	5246	55	78	646	85	859	110	743	95
30	5049	59	5334	60	5191	59	79	561	80	749	100	648	90
31	4488	60	5274	60	5132	60	80	481	75	649	95	558	90
32	4928	60	5214	65	5072	62	81	406	70	554	90	468	84
33	4868	60	5142	65	5010	63	82	336	65	464	85	384	75
34	4808	60	5084	65	4947	63	83	271	60	379	80	309	65
35	4748	60	5019	60	4884	59	84	211	50	299	75	244	55
36	4688	60	4959	56	4825	58	85	161	40	224	55	189	45
37	4628	60	4903	56	4767	58	86	121	30	169	40	144	35
38	4568	60	4847	56	4709	58	87	91	22	129	30	109	27
39	4508	60	4797	58	4651	60	88	69	17	99	23	82	20
40	4448	65	4733	65	4591	65	89	52	14	76	18	62	15
41	4383	72	4668	75	4526	73	90	38	12	58	15	47	14
42	4311	80	4593	76	4453	78	91	26	9	43	12	33	12
43	4231	80	4517	76	4375	78	92	17	7	31	10	21	10
44	4151	80	4441	75	4297	78	93	10	6	21	8	11	6
45	4071	80	4366	72	4219	76	94	4	3	13	6	5	3
46	3991	80	4294	67	4143	74	95	1	1	7	4	2	1
47	3911	80	4227	65	4069	72	96	0	0	3	2	1	1
48	3831	80	4162	65	3997	73	97	0	0	1	1	0	0

Age.	Des Parcieux.	Sweden.	Northampton.	Age.	Des Parcieux.	Sweden.	Northampton.
1		42·95	32·74	49	21·07	19·09	18·49
2		44·92	37·79	50	20·38	18·46	17·99
3	47·71	46·11	39·55	51	19·73	17·87	17·50
4	48·17	46·78	40·58	52	19·11	17·29	17·02
5	48·27	46·79	40·84	53	18·48	16·70	16·54
6	48·20	46·66	41·07	54	17·85	16·12	16·06
7	47·98	46·43	41·03	55	17·25	15·53	15·58
8	47·66	46·07	40·79	56	16·64	14·95	15 10
9	47·30	45·61	40·36	57	16·02	14·37	14·63
10	46·83	45·07	39·78	58	15·44	13·79	14·15
11	46·26	44·38	39·14	59	14·84	13·21	13·68
12	45·58	43·70	38·49	60	14·25	12·63	13·21
13	44·89	43·01	37·83	61	13·65	12·12	12·75
14	44·20	42·33	37·17	62	13·04	11·62	12·28
15	43·51	41·64	36·51	63	12·43	11·11	11·81
16	42·82	40·92	35·85	64	11·86	10·61	11·35
17	42·17	40·19	35·20	65	11·26	10·10	10·88
18	41·52	39·47	34·58	66	10·69	9·62	10·42
19	40·87	38·74	33·99	67	10·14	9·15	9·96
20	40·22	38·02	33·45	68	9·61	8·67	9·50
21	39·62	37·33	32·90	69	9·11	8·20	9·05
22	39·00	36·64	32·30	70	8·64	7·72	8·60
23	38·40	35·96	31·88	71	8·17	7·32	8·17
24	37·78	35·27	31·36	72	7·73	6·89	7·74
25	37·17	34·58	30·85	73	7·31	6·53	7·33
26	36·55	33·91	30·33	74	6·90	6·23	6·92
27	35·93	33·23	29·82	75	6·50	5·91	6·54
28	35·30	32·56	29·30	76	6·10	5·59	6·18
29	34·69	31·88	28·79	77	5·71	5·28	5·83
30	34·06	31·21	28·27	78	5·36	4·96	5·48
31	33·29	30·57	27·76	79	5·00	4·61	5·11
32	32·80	29·94	27·24	80	4·69	4·28	4·75
33	32·16	29·30	26·72	81	4·39	4·01	4·41
34	31·52	28·67	26·20	82	4·01	3·80	4·09
35	30·88	28·03	25·68	83	3·84	3·57	3·80
36	30·23	27·31	25·16	84	3·52	3·39	3·58
37	29·58	26·68	24·64	85	3·21	3·23	3·37
38	28·89	26·01	24·12	86	2·92	3·09	3·19
39	28·18	25·33	23·60	87	2·67	2·92	3·01
40	27·48	24·66	23·08	88	2·36	2·71	2·86
41	26·77	24·05	22·56	89	2·06	2·43	2·66
42	26·06	23·44	22·04	90	1·77	2·05	2·41
43	25·34	22·83	21·54	91	1·50	1·71	2·09
44	24·62	22·22	21·03	92	1·25	1·40	1·75
45	23·89	21·61	20·52	93	1·00	1·23	1·37
46	23·15	20·98	20·02	94		1·10	1·05
47	22·45	20·35	19·51	95		1·00	·75
48	21·74	19·72	19·00	96			·50

Age.	Sweden, 4 per Cent.		Northampton.		Des Parcieux, 4½ per cent.	Age.	Sweden, 4 per Cent.		Northampton.		Des Parcieux, 4½ per cent.
	MALES.	FEMALES.	4 PER CENT.	5 PER CENT.			MALES.	FEMALES.	4 PER CENT.	5 PER CENT.	
1	16·508	16·820	13·465	11·563		49	11·528	12·333	11·475	10·443	12·176
2	17·355	17·719	15·633	13·420		50	11·267	12·049	11·264	10·269	11·921
3	17·935	18·344	16·462	14·135	16·756	51	11·030	11·769	11·057	10·097	11·675
4	18·328	18·780	17·010	14·613	17·052	52	10·785	11·492	10·849	9·925	11·440
5	18·503	18·927	17·248	14·827	17·233	53	10·531	11·220	10·637	9·748	11·195
6	18·622	19·045	17·482	15·041	17·357	54	10·269	10·937	10·421	9·567	10·938
7	18·693	19·131	17·611	15·166	17·435	55	9·998	10·642	10·201	9·382	10·691
8	18·725	19·162	17·662	15·226	17·482	56	9·717	10·334	9·977	9·193	10·433
9	18·715	19·151	17·625	15·210	17·515	57	9·425	10·012	9·749	8·999	10·163
10	18·674	19·109	17·523	15·139	17·512	58	9·440	9·692	9·516	8·801	9·902
11	18·600	19·041	17·393	15·043	17·468	59	8·845	9·358	9·280	8·599	9·631
12	18·491	18·952	17·251	14·937	17·380	60	8·540	9·039	9·039	8·392	9·346
13	18·378	18·840	17·103	14·826	17·289	61	8·241	9·739	8·795	8·181	9·049
14	18·246	18·707	16·950	14·710	17·194	62	7·950	9·453	8·547	7·966	8·738
15	18·105	18·568	16·791	14·588	17·095	63	7·669	9·166	8·291	7·742	8·433
16	17·958	18·424	16·625	14·460	16·991	64	7·382	7·870	8·030	7·514	8·114
17	17·803	18·290	16·462	14·334	16·905	65	7·090	7·566	7·761	7·276	7·780
18	17·643	18·151	16·309	14·217	16·815	66	6·792	7·252	7·488	7·034	7·451
19	17·492	18·013	16·167	14·108	16·721	67	6·489	6·930	7·211	6·787	7·129
20	17·335	17·872	16·033	14·007	16·624	68	6·201	6·596	6·930	6·536	6·814
21	17·192	17·725	15·912	13·917	16·544	69	5·933	6·253	6·647	6·281	6·511
22	17·042	17·573	15·797	13·833	16·462	70	5·670	5·897	6·361	6·023	6·221
23	16·887	17·414	15·680	13·746	16·377	71	5·418	5·564	6·075	5·764	5·925
24	16·742	17·252	15·560	13·658	16·289	72	5·180	5·261	5·790	5·504	5·648
25	16·592	17·087	15·438	13·567	16·198	73	4·940	4·998	5·507	5·245	5·373
26	16·436	16·915	15·312	13·473	16·104	74	4·724	4·792	5·230	4·990	5·101
27	16·274	16·751	15·184	13·377	16·006	75	4·487	4·582	4·962	4·744	4·836
28	16·105	16·588	15·053	13·278	15·905	76	4·253	4·367	4·710	4·511	4·553
29	15·930	16·427	14·918	13·177	15·800	77	4·024	4·145	4·457	4·277	4·281
30	15·751	16·261	14·781	13·072	15·691	78	3·768	3·913	4·197	4·035	4·025
31	15·575	16·104	14·639	12·965	15·578	79	3·512	3·668	3·921	3·776	3·763
32	15·395	15·941	14·495	12·854	15·460	80	3·260	3·402	3·643	3·515	3·533
33	15·208	15·787	14·347	12·740	15·338	81	3·017	3·145	3·377	3·263	3·313
34	15·014	15·629	14·195	12·623	15·211	82	3·792	2·905	3·122	3·020	3·114
35	14·812	15·465	14·039	12·502	15·078	83	3·600	2·699	2·887	2·797	2·895
36	14·601	15·278	13·880	12·377	14·941	84	3·473	2·559	2·708	2·627	2·641
37	14·382	15·070	13·716	12·249	14·797	85	3·371	2·552	2·543	2·471	2·392
38	14·154	14·854	13·548	12·116	14·624	86	3·281	2·518	2·393	2·328	2·158
39	13·916	14·629	13·375	11·979	14·444	87	3·154	2·431	2·251	2·193	1·955
40	13·668	14·401	13·197	11·837	14·254	88	1·955	2·294	2·131	2·080	1·693
41	13·426	14·185	13·018	11·695	14·056	89	1·698	2·108	1·967	1·924	1·433
42	13·196	13·994	12·838	11·551	13·849	90	1·417	1·873	1·758	1·723	1·178
43	12·984	13·798	12·657	11·407	13·631	91	1·154	1·628	1·474	1·447	0·934
44	12·763	13·596	12·472	11·258	13·403	92	0·835	1·349	1·171	1·153	0·707
45	12·535	13·383	12·283	11·105	13·164	93	0·477	1·071	0·827	0·816	0·478
46	12·297	13·151	12·089	10·947	12·913	94	0·240	0·799	0·530	0·524	
47	12·051	12·894	11·890	10·784	12·672	95		0·544	0·240	0·238	
48	11·795	12·620	11·685	10·616	12·419	96		0·320			

TABLE XI. ANNUITIES ON TWO JOINT LIVES.—DES PARCIEUX, 4½ PER CENT.

Age.	Diff. 0.	Diff. 5.	Diff. 10.	Diff. 20.	Diff. 30.	Diff. 40.	Diff. 50.	Diff. 60.
3	13·571	14·219	13·535	13·566	12·893	11·665	9·770	7.520
4	14·072	14·525	14·345	13·765	13·048	11·706	9·741	7·370
5	14·396	14·708	14·446	13·870	13·112	11·658	9·653	7·169
6	14·632	14·811	14·498	13·929	13·130	11·568	9·522	6·938
7	14·796	14·840	14·528	13·951	13·111	11·440	9·356	6·693
8	14·910	14·841	14·531	13·946	13·043	11·289	9·181	6·443
9	15·004	14·828	14·521	13·929	12·960	11·138	8·989	6·197
10	15·038	14·783	14·479	13·882	12·843	10·958	8·767	5·951
11	15·004	14·701	14·421	13·801	12·690	10·754	8·513	5·685
12	14·897	14·598	14·326	13·684	12·498	10·538	8·224	5·424
13	14·786	14·491	14·228	13·562	12·296	10·311	7·941	5·162
14	14·669	14·380	14·126	13·434	12·084	10·073	7·643	4·903
15	14·547	14·264	14·020	13·301	11·860	9·843	7·328	4·649
16	14·419	14·161	13·909	13·162	11·624	9·601	7·017	4·377
17	14·321	14·071	13·811	13·033	11·410	9·359	6·719	4·119
18	14·220	13·979	13·710	12·878	11·185	9·125	6·428	3·876
19	14·114	13·884	13·605	12·716	10·968	8·879	6·145	3·625
20	14·004	13·784	13·496	12·545	10·739	8·621	5·873	3·404
21	13·926	13·699	13·400	12·382	10·532	8·362	5·603	3·197
22	13·846	13·612	13·301	12·211	10·335	8·088	5·351	3·010
23	13·764	13·522	13·199	12·032	10·128	7·820	5·098	2·804
24	13·679	13·429	13·092	11·843	9·911	7·538	4·848	2·562
25	13·591	13·333	12·982	11·645	9·702	7·241	4·604	2·325
26	13·501	13·233	12·868	11·436	9·484	6·948	4·342	2·100
27	13·408	13·131	12·749	11·236	9·255	6·659	4·089	1·907
28	13·312	13·024	12·606	11·025	9·034	6·378	3·851	1·654
29	13·213	12·914	12·455	10·823	8·803	6·105	3·607	1·403
30	13·110	12·799	12·298	10·611	8·561	5·845	3·391	1·155
31	13·004	12·680	12·133	10·407	8·306	5·578	3·186	0·918
32	12·893	12·556	11·960	10·212	8·038	5·329	3·000	0·697
33	12·779	12·408	11·778	10·008	7·775	5·081	2·796	0·473
34	12·660	12·252	11·587	9·794	7·499	4·835	2·556	
35	12·536	12·089	11·385	9·589	7·207	4·595	2·320	
36	12·408	11·918	11·174	9·374	6·920	4·338	2·099	
37	12·274	11·739	10·970	9·148	6·639	4·091	1·908	
38	12·095	11·531	10·378	8·916	6·353	3·852	1·655	
39	11·907	11·313	10·512	8·673	6·076	3·607	1·403	
40	11·710	11·082	10·274	8·417	5·811	3·390	1·156	
41	11·502	10·839	10·042	8·147	5·538	3·184	0·918	
42	11·282	10·601	9·817	7·862	5·282	2·996	0·697	
43	11·051	10·349	9·579	7·582	5·025	2·790	0·473	
44	10·807	10·102	9·329	7·285	4·770	2·547		
45	10·549	9·841	9·083	6·972	4·518	2·309		
46	10·276	9·583	8·824	6·659	4·248	2·082		
47	10·023	9·345	8·566	6·360	3·991	1·888		
48	9·756	9·095	8·312	6·065	3·748	1·636		

TABLE XI. ANNUITIES ON TWO JOINT LIVES.—DES PARCIEUX, 4½ PER CENT.

Age.	Diff. 0.	Diff. 5.	Diff. 10.	Diff. 20.	Diff. 30.	Diff. 40.
49	9·508	8·846	8·060	5·787	3·503	1·387
50	9·246	8·602	7·793	5·517	3·284	1·141
51	9·004	8·360	7·526	5·250	3·080	0·906
52	8·782	8·120	7·257	5·007	2·899	0·689
53	8·549	7·886	6·992	4·762	2·700	0·469
54	8·303	3·639	6·711	4·518	2·464	
55	8·077	7·394	6·428	4·286	2·238	
56	7·839	7·136	6·145	4·037	2·023	
57	7·588	6·862	5·864	3·794	1·836	
58	7·357	6·605	5·598	3·573	1·596	
59	7·114	6·333	5·339	3·343	1·356	
60	6·857	6·045	5·088	5·140	1·119	
61	6·585	5·757	4·828	2·944	0·891	
62	6·297	9·467	4·578	2·764	0·677	
63	6·024	5·191	4·337	2·573	0·463	
64	5·733	4·917	4·093	2·346		
65	5·422	4·647	3·848	2·120		
66	5·123	4·377	3·594	1·909		
67	4·835	4·128	3·351	1·729		
68	4·560	3·885	3·126	1·498		
69	4·300	3·653	2·902	1·270		
70	4·062	3·436	2·709	1·049		
71	3·817	3·203	2·524	0·834		
72	3·599	2·989	2·365	0·638		
73	3·384	2·788	2·195	0·440		
74	3·176	2·585	1·999			
75	2·977	2·409	1·811			
76	2·758	2·232	1·627			
77	2·549	2·076	1·473			
78	2·362	1·917	1·279			
79	2·165	1·730	1·082			
80	2·005	1·562	0·895			
81	1·860	1·408	0·717			
82	1·745	1·292	0·559			
83	1·614	1·130	0·398			
84	1·442	0·454				
85	1·276	0·782				
86	1·128	0·621				
87	1·024	0·489				
88	0·860	0·348				
89	0·699					
90	0·545					
91	0·405					
92	0·296					
93	0·239					
94	0·000					

SWEDEN, 4 PER CENT.

Age.	Diff. 0.	Diff. 6.	Diff. 12.	Diff. 18.	Diff. 24.	Diff. 30.	Diff. 36.	Diff. 42.
1	12·252	13·989	13·894	13·389	12·832	12·196	11·465	10·546
2	13·583	14·780	14·557	14·008	13·409	12·730	11·913	10·946
3	14·558	15·323	14·988	14·417	13·778	13·066	12·164	11·168
4	15·267	15·685	15·259	14·671	14·003	13·264	12·284	11·260
5	15·577	15·817	15·326	14·725	14·037	13·277	12·242	11·183
6	15·820	15·887	15·354	14·740	14·033	13·242	12·185	11·064
7	16·003	15·914	15·351	14·727	14·006	13·170	12·112	10·915
8	16·109	15·888	15·310	14·673	13·994	13·059	12·004	10·743
9	16·152	15·824	15·244	14·590	13·855	12·923	11·865	10·560
10	16·141	15·729	15·149	14·484	13·741	12·743	11·694	10·357
11	16·087	15·617	15·033	14·357	13·604	12·563	11·493	10·140
12	15·982	15·477	14·889	14·202	13·428	12·379	11·259	9·898
13	15·855	15·327	14·736	14·045	13·234	12·196	11·011	9·644
14	15·701	15·164	14·566	13·874	13·023	11·997	10·759	9·371
15	15·535	15·001	14·392	13·700	12·798	11·787	10·514	9·087
16	15·361	14·832	14·216	13·520	12·570	11·562	10·264	8·799
17	15·196	14·665	14·042	13·340	12·351	11·328	10·018	8·503
18	15·023	14·491	13·860	13·141	12·146	11·076	9·761	8·208
19	14·854	14·320	13·687	12·934	11·951	10·819	9·500	7·928
20	14·682	14·144	13·512	12·720	11·751	10·567	9·228	7·658
21	14·525	13·976	13·345	12·505	11·550	10·332	8·953	7·396
22	14·360	13·807	13·173	12·286	11·335	10·092	8·675	7·127
23	14·194	13·635	12·997	12·073	11·107	9·852	8·385	6 851
24	14·020	13·455	12·801	11·873	10·862	9·602	8·097	6·566
25	13·849	13·284	12·599	11·683	10·612	9·347	7·823	6·275
26	13·671	13·108	12·387	11·485	10·364	9·080	7·557	5·986
27	13·495	12·935	12·170	11·284	10·130	8·807	7·297	5·702
28	13·323	12 763	11·953	11·072	9·894	8·534	7·032	5·415
29	13·148	12·586	11·742	10·847	9·659	8·250	6·761	5·136
30	12·965	12·390	11·543	10·606	9·413	7·967	6·481	4·881
31	12·795	12·292	11·359	10·365	9·167	7·702	6·197	4·646
32	12·624	11·398	11·170	10·128	8·912	7·446	5·917	4·453
33	12·456	11·779	10·978	9·905	8·651	7·196	5·642	4·251
34	12·286	11·586	10·775	9·679	8·389	6·942	5·364	4·040
35	12·109	11·361	10·557	9·452	8·114	6·679	5·093	3·833
36	11·904	11.156	10·314	9·207	7·833	6·402	4·840	3·605
37	11·683	10·953	10·059	8·951	7·561	6·115	4·603	3·352
38	11·452	10·741	9·805	8·683	7·296	5·828	4·405	3·098
39	11·209	10·519	9·558	8·404	7·033	5·543	4·195	2·889
40	10·964	10·286	9·308	8·124	6·763	5·254	3·975	2·710
41	10·732	10·049	9·066	7·839	6·492	4·977	3·762	2·553
42	10·531	9·813	8·830	7·569	6·225	4·730	3·539	2·418
43	10·346	9·581	8·597	7·318	5·957	4·507	3·295	2·305
44	10·154	9·351	8·354	7·075	5·689	4·322	3·052	2·203
45	9·995	9·129	8·101	6·836	5·426	4·128	2·854	2·083
46	9·736	8·897	7·841	6·586	5·153	3·921	2·684	1·933
47	9·497	8·658	7·563	6·323	4·884	3·715	2·533	1·708
48	9·236	8·402	7·281	6·048	4·633	3·489	2·396	1·385

SWEDEN, 4 PER CENT.

Age.	Diff. 0.	Diff. 6.	Diff. 12.	Diff. 18.	Diff. 24.	Diff. 30.	Diff. 36.	Diff. 42.
49	8·966	8·139	7·008	5·764	4·398	3·238	2·277	1·090
50	8·707	7·874	6·749	5·487	4·205	2·990	2·171	0·818
51	8·469	7·613	6·505	5·221	4·008	2·792	2·050	0·862
52	8·230	7·351	6·256	4·953	3·803	2·623	1·901	0·551
53	7·994	7·083	6·004	4·694	3·605	2·475	1·681	0·468
54	7·748	6·814	5·743	4·455	3·389	2·344	1·366	
55	7·495	6·555	5·474	4·231	3·150	2·232	1·078	
56	7·229	6·299	5·204	4·043	2·909	2·130	0·810	
57	6·954	6·045	4·936	3·844	2·710	2·010	0·655	
58	6·678	5·788	4·664	3·637	2·539	1·864	0·546	
59	6·388	5·519	4·395	3·430	2·385	1·644	0·464	
60	6·104	5·249	4·149	3·210	2·248	1·333		
61	5·844	4·984	3·927	3·974	2·135	1·050		
62	5·600	4·729	3·747	3·744	2·037	0·789		
63	5·367	4·482	3·563	3·557	1·926	0·639		
64	5·128	4·231	3·370	3·396	1·790	0·533		
65	4·881	3·982	3·180	3·252	1·585	0·456		
66	4·626	3·750	2·974	3·123	1·290			
67	4·362	3·527	2·743	3·010	1·017			
68	4·103	3·340	2·514	1·910	0·764			
69	3·851	3·147	2·324	1·798	0·617			
70	3·593	2·946	2·155	1·661	0·514			
71	3·345	2·752	2·004	1·464	0·441			
72	3·128	2·558	1·875	1·189				
73	2·935	2·355	1·768	0·937				
74	2·797	2·172	1·692	0·708				
75	2·648	2·017	1·605	0·575				
76	2·490	1·877	1·497	0·481				
77	2·340	1·756	1·339	0·421				
78	2·170	1·639	1·097					
79	1·967	1·524	0·863					
80	1·758	1·416	0·638					
81	1·600	1·320	0·511					
82	1·472	1·225	0·427					
83	1·364	1·094	0·379					
84	1·276	0·902						
85	1·212	0·725						
86	1·172	0·556						
87	1·127	0·459						
88	1·071	0·396						
89	0·949	0·364						
90	0·718							
91	0·516							
92	0·326							
93	0·236							
94	0·190							
95	0·024							
96	0·000							

TABLE XIII. ANNUITIES ON TWO JOINT LIVES.
NORTHAMPTON, 4 PER CENT.

Age.	Diff. 0.	Diff. 5.	Diff. 10.	Diff. 15.	Diff. 20.	Diff. 25.	Diff. 30.	Diff. 35.
1	8·252	10·741	10·782	10·406	10·053	9·770	9·438	9·047
2	11·107	12·581	12·438	11·981	11·605	11·264	10·865	10·392
3	12·325	13·319	13·019	12·531	12·161	11·790	11·355	10·838
4	13·185	13·775	13·374	12·876	12·511	12·116	11·651	11 097
5	13·591	13·933	13·479	12·993	12·633	12·220	11·732	11·150
6	14·005	14·068	13·578	13·121	12·754	12·322	11·812	11·203
7	14·224	14·111	13·599	13·178	12·798	12·350	11·819	11·190
8	14.399	14·089	13·569	13·178	12·786	12·323	11·772	11·130
9	14·396	13·992	13·482	13·112	12·710	12·234	11·665	11·012
10	14·277	13·841	13·355	12·998	12·586	12·098	11·513	10·851
11	14·133	13·664	13·217	12·861	12·441	11·941	11·342	10·697
12	13·966	13·480	13·078	12·715	12·286	11·773	11·165	10·481
13	13·789	13·303	12·934	12·564	12·125	11·600	10·985	10·284
14	13·604	13·130	12·784	12·408	11·959	11·420	10·799	10·080
15	13·411	12·961	12·630	12·246	11·787	11·234	10·607	9·872
16	13·212	12·799	12·470	12·078	11·609	11·044	10·408	9·665
17	13·019	12·646	12·311	11·911	11·430	10·856	10·208	9·461
18	12·841	12·500	12·158	11·750	11·257	10 677	10·011	9·260
19	12·679	12·361	12·013	11·595	11·089	10·502	9·818	9·063
20	12·535	12·229	11·873	11·445	10·924	10·330	9·630	8·869
21	12·409	12·105	11·742	11·302	10·768	10·165	9·454	8·679
22	12·293	11·987	11·615	11·163	10·619	10·001	9·284	8·491
23	12·179	11·866	11·485	11·020	10·470	9·833	9·111	8·299
24	12·062	11·743	11·352	10·874	10·317	9·661	8·934	8·104
25	11·944	11·618	11·217	10·725	10·160	9·488	8·754	7·906
26	11·822	11·489	11·078	10·574	10·000	9·318	8·570	7·704
27	11·699	11·359	10·936	10·423	9·836	9·148	8·383	7·499
28	11·573	11·225	10·791	10·272	9·667	8·975	8·193	7·286
29	11·445	11·088	10·642	10·117	9·495	8·799	7·999	7·069
30	11·313	10·948	10·490	9·959	9·321	8·619	7·802	6·844
31	11·179	10·805	10·336	9·797	9·151	8·436	7·601	6·615
32	11·042	10·659	10·182	9·631	8·980	8·250	7·397	6·382
33	10·902	10·508	10·027	9·461	8·806	8·060	7·186	6·146
34	10·759	10·354	9·869	9·286	8·629	7·866	6·671	5·906
35	10·612	10·196	9·706	9·110	8·448	7·669	6·747	5·663
36	10·462	10·037	9·540	8·937	8·264	7·469	6·520	5·419
37	10·307	9·877	9·370	8·763	8·076	7·265	6·288	5·174
38	10·149	9·716	9·195	8·586	7·884	7·053	6·052	4·930
39	9·986	9·550	9·015	8·406	7·689	6·838	5·813	4·690
40	9·820	9·381	8·834	8·221	7·490	6·614	5·571	4·457
41	9·654	9·210	8·658	8·035	7·290	6·388	5·329	4·238
42	9·491	9·037	8·483	7·848	7 088	6·159	5·087	4·019
43	9·326	8·862	8·308	7·660	6·881	5·929	4·848	3·794
44	9·160	8·683	8·130	7·469	6·671	5·696	4·613	3·552
45	8·990	8·503	7·948	7·274	6·453	5·460	4·386	3·308
46	8·815	8·326	7·763	7·076	6·230	5·220	4·171	3·072
47	8·637	8·147	7·574	6·875	6·004	4·983	3·954	2·843
48	8·453	7·965	7·382	6·667	5·774	4·746	3·731	2·632

ANNUITIES ON TWO JOINT LIVES.
NORTHAMPTON, 4 PER CENT.

Age.	Diff. 0.	Diff. 5.	Diff. 10.	Diff. 15.	Diff. 20.	Diff. 25.	Diff. 30.	Diff. 35.
49	8·266	7·780	7·186	6·454	5·541	4·511	3·490	2·470
50	8·081	7·593	6·989	6·236	5·306	4·285	3·247	2·322
51	7·900	7·409	6·795	6·019	5·074	4·074	3·015	2·188
52	7·724	7·225	6·600	5·801	4·845	3·864	2·792	2·063
53	7·544	7·039	6·399	5·580	4·614	3·648	2·585	1·960
54	7·362	6·850	6·196	5·357	4·389	3·416	2·428	1·817
55	7·179	6·659	5·986	5·132	4·171	3·180	2·284	1·633
56	6·993	6·465	5·774	4·905	3·966	2·953	2·153	1·377
57	6·805	6·270	5·559	4·679	3·761	2·733	2·030	1·102
58	6·614	6·070	5·341	4·455	3·549	2·530	1·928	0·784
59	6·421	5·867	5·121	4·234	3·322	2·376	1·788	0·505
60	6·226	5·658	4·900	4·021	3·092	2·234	1·608	0·230
61	6·030	5 447	4·679	3·821	2·870	2·105	1·358	
62	5·831	5·285	4·458	3·621	2·656	1·985	1·088	
63	5·626	5 017	4·236	3·414	2·457	1·886	0·774	
64	5·417	4·798	4·019	3·192	2·305	1·751	0·500	
65	5·201	4·573	3·806	2·965	2·163	1·575	0·228	
66	4·982	4·349	3·606	2·746	2·035	1·330		
67	4·760	4·124	3·405	2·533	1·915	1·067		
68	4·537	3·901	3·199	2·336	1·817	0·760		
69	4·312	3 683	2·979	2·183	1·685	0·491		
70	4·087	3·471	2·757	2·042	1·515	0·224		
71	3·862	3·270	2·542	1·914	1·280			
72	3·636	3 070	2·334	1·794	1·028			
73	3·421	2·869	2·141	1·697	0·733			
74	3·211	2·659	1·991	1·570	0·474			
75	3·015	2·448	1·856	1·413	0·217			
76	2·833	2·258	1·737	1·200				
77	2·656	2·077	1·633	0·970				
78	2·470	1·899	1·546	0·697				
79	2·271	1·751	1·427	0·453				
80	2·068	1·608	1·278	0·208				
81	1·869	1·478	1·078					
82	1·681	1·356	0·864					
83	1·510	1·259	0·614					
84	1·387	1·164	0·403					
85	1·339	1·054	0·187					
86	1·195	0·902						
87	1·124	0·738						
88	1·030	0·554						
89	1·015	0·373						
90	0.922	0·177						
91	0·756							
92	0·583							
93	0·365							
94	0·201							
95	0·060							
96	0·000							

TABLE XIII. ANNUITIES ON TWO JOINT LIVES.
NORTHAMPTON, 4 PER CENT.

Age.	Diff. 40.	Diff. 45.	Diff. 50.	Diff. 55.	Diff. 60.	Diff. 65.	Diff. 70.
1	8·585	8·071	7·479	6·843	6·123	5·295	4·380
2	9·839	9·221	8·520	7·756	6·894	5·896	4·814
3	10·242	9·566	8·815	7·986	7·048	5·965	4·811
4	10·468	9·744	8·957	8·075	7·076	5·924	4·726
5	10·500	9·742	8·931	8·011	6·963	5·768	4·557
6	10·528	9 745	8·902	7·944	6·846	5·610	4·403
7	10·491	9·690	8·817	7·828	6·684	5·418	4·222
8	10·404	9·591	8·691	7·669	6·490	5·204	4·106
9	10·263	9·442	8·519	7·470	6·262	4·969	3·775
10	10·085	9·256	8·314	7·236	6·008	4·725	3·517
11	9·894	9·052	8·092	6·987	5·744	4·487	3·264
12	9·698	8·839	7·863	6·730	5·478	4·368	3·020
13	9·497	8·622	7·625	6·468	5·212	4·022	2·794
14	9·290	8·399	7·381	6·202	4·950	3·759	2·622
15	9·077	8·170	7·127	5·933	4·695	3·492	2·462
16	8·858	7·935	6·866	5·660	4·452	3·235	2·315
17	8·639	7·700	6·604	5·389	4·210	2·987	2·177
18	8·422	7·462	6·343	5·123	3·964	2·760	2·061
19	8·207	7·226	6·084	4·866	3·704	2·589	1·904
20	7·995	6·986	5·826	4·619	3·443	2·431	1·704
21	7·787	6·749	5·572	4·391	3·195	2·290	1·432
22	7·580	6·512	5·321	4·164	2·958	2·158	1·142
23	7·365	6·271	5 072	3·930	2·740	2·048	0·809
24	7·147	6·027	4·827	3·679	2·574	1·895	0·520
25	6·920	5·780	4·589	3·425	2·421	1·699	0·236
26	6·689	5·532	4·365	3·181	2·282	1·429	
27	6·454	5·283	4·140	2·945	2·151	1·140	
28	6·215	5·036	3·908	2·728	2·041	0·808	
29	5·973	4·792	3·659	2·563	1·889	0·519	
30	5·729	4·557	3·406	2·411	1·694	0·236	
31	5·483	4·335	3·164	2·272	1·425		
32	5·236	4·111	2·929	2·142	1·137		
33	4·991	3·881	2·713	2·033	0·806		
34	4·749	3·633	2·549	1·882	0·518		
35	4·516	3·383	2·398	1·688	0·235		
36	4·295	3·142	2·260	1·420			
37	4·173	2·909	2·130	1·134			
38	3·844	2·694	2·022	0·804			
39	3·598	2·530	1·872	0·517			
40	3·349	2·379	1·679	0·235			
41	3·109	2·241	1·413				
42	2·878	2·113	1·128				
43	2·666	2·006	0·800				
44	2·505	1·859	0·515				
45	2·356	1·668	0·234				
46	2·221	1·405					
47	2·093	1·122					
48	1·987	1·797					

Diff. 40.—*Continued.*

49	1·840
50	1·651
51	1·391
52	1·113
53	0·790
54	0·509
55	0·232

NORTHAMPTON, 4 PER CENT.

Ages of A. B. C.	Diff. 0.	Ages of B. C.	Diff. 10 and 20.
1	5·309	11 21	8·627
2	8·251	12 22	9·914
3	9·632	13 23	10·344
4	10·661	14 24	10·598
5	11·170	15 25	10·655
6	11·707	16 26	10·708
7	12·058	17 27	10·700
8	12·266	18 28	10·654
9	12·298	19 29	10·562
10	12·200	20 30	10·438
11	12·043	21 31	10·305
12	11·865	22 32	10·170
13	11·678	23 33	10·031
14	11·481	24 34	9·887
15	11·274	25 35	9·738
16	11·056	26 36	9·584
17	10·845	27 37	9·429
18	10·656	28 38	9·278
19	10·490	29 39	9·131
20	10·342	30 40	8·986
21	10·222	31 41	8·850
22	10·118	32 42	8·718
23	10·012	33 43	8·586
24	9·905	34 44	8·451
25	9·796	35 45	8·313
26	9·685	36 46	8·171
27	9·572	37 47	8·027
28	9·457	38 48	7·878
29	9·340	39 49	7·725
30	9·221	40 50	7·571
31	9·099	41 51	7·420
32	8·975	42 52	7·272
33	8·848	43 53	7·123
34	8·718	44 54	6·971
35	8·585	45 55	6·816
36	8·448	46 56	6·658
37	8·309	47 57	6·497
38	8·165	48 58	6·332
39	8·017	49 59	6·164
40	7·865	50 60	5·994
41	7·714	51 61	5·827
42	7·567	52 62	5·662
43	7·423	53 63	5·494
44	7·276	54 64	5·322
45	7·126	55 65	5·145
46	6·972	56 66	4·965
47	6·813	57 67	4·782
48	6·650	58 68	4·597

Ages of A. B. C.	Diff. 0.	Ages of B. C.	Diff. 10 and 20.
49	6·482	59 69	4·408
50	6·317	60 70	4·219
51	6·161	61 71	4·032
52	6·011	62 72	3·847
53	5·859	63 73	3·660
54	5·705	64 74	3·477
55	5·550	65 75	3·298
56	5·393	66 76	3·128
57	5·235	67 77	2·959
58	5·076	68 78	2·785
59	4·916	69 79	2·598
60	4·755	70 80	2·408
61	4·593	71 81	2·224
62	4·432	72 82	2·044
63	4·263	73 83	1·875
64	4·093	74 84	1·743
65	3·914	75 85	1·623
66	3·733	76 86	1·519
67	3·550	77 87	1·425
68	3·366	78 88	1·350
69	3·181	79 89	1·248
70	2·995	80 90	1·122
71	2·810	81 91	0·951
72	2·627	82 92	0·767
73	2·448	83 93	0·548
74	2·277	84 94	0·362
75	2·119	85 95	0·119
76	1·985		
77	1·855		
78	1·720		
79	1·563		
80	1·400		
81	1·245		
82	1·092		
83	0·949		
84	0·860		
85	0·782		
86	0·716		
87	0·662		
88	0·646		
89	0·614		
90	0·563		
91	0·452		
92	0·337		
93	0·185		
94	0·085		
95	0·015		

TABLE XV. LOGARITHMS OF D, N, AND M.

ENGLISH, 3 PER CENT., MALES.

Age.	λD	λN	λM	Age.	λD	λN	λM
0	4·7098992	5·9935779	4·3536342	53	3·6655131	4·8014451	3·4448852
1	4·6216760	5·9703689	4·1656386	54	3·6446469	4·7684661	3·4318622
2	4·5805823	5·9504618	4·0821819	55	3·6235767	4·7345169	3·4187358
3	4·5527516	5·9315236	4·0346066	56	3·6014916	4·6995036	3·4042617
4	4·5291694	5·9129775	3·9992105	57	3·5784033	4·6633870	3·3884824
5	4·5085060	5·8946498	3·9729488	58	3·5545297	4·6261222	3·3717973
6	4·4893367	5·8764225	3·9514448	59	3·5297918	4·5876371	3·3540881
7	4·4713747	5·8582355	3·9340414	60	3·5041091	4·5478559	3·3352563
8	4·4543856	5·8400390	3·9200094	61	3·4773979	5·5066979	3·3152574
9	4·4381957	5·8217907	3·9087398	62	3·4495706	4·4640773	3·2939657
10	4·4226398	5·8034538	3·8997561	63	3·4205347	4·4199022	3·2712706
11	4·4076049	5·7849965	3·8926236	64	3·3901917	4·3740743	3·2471144
12	4·3925701	5·7663901	3·8855852	65	3·3584364	4·3264885	3·2213863
13	4·3775353	5·7476267	3·8786799	66	3·3251554	4·2770317	3·1939512
14	4·3619794	5·7286979	3·8703237	67	3·2902265	4·2255825	3·1647274
15	4·3461930	5·7096183	3·8613640	68	3·2535176	4·1720103	3·1335274
16	4·3303185	5·6903900	3·8522949	69	3·2148856	4·1161749	3·1002111
17	4·3143652	5·6710095	3·8431541	70	3·1741754	4·0579251	3·0646526
18	4·2983311	5·6514719	3·8339045	71	3·1312192	3·9970985	3·0266992
19	4·2822141	5·6317723	3·8245466	72	3·0858350	3·9335207	2·9861643
20	4·2660120	5·6119059	3·8151169	73	3·0378261	3·8670036	2·9427654
21	4·2497227	5·5918670	3·8055790	74	2·9869797	3·7973457	2·8963973
22	4·2333439	5·5716504	3·7959691	75	2·9330662	3·7243302	2·8468250
23	4·2168732	5·5512500	3·7862153	76	2·8758380	3·6477247	2·7937019
24	4·2003083	5·5306600	3·7764240	77	2·8150285	3·5672797	2·7369350
25	4·1836465	5·5098732	3·7664888	78	2·7503513	3·4827278	2·6760448
26	4·1668855	5·4888833	3·7564441	79	2·6814991	3·3937828	2·6108506
27	4·1500224	5·4676829	3·7463246	80	2·6081425	3·3001384	2·5409170
28	4·1330547	5·4462643	3·7360944	81	2·5299293	3·2014672	2·4660546
29	4·1159795	5·4246196	3·7257530	82	2·4464832	3·0974196	2·3858057
30	4·0987941	5·4027401	3·7153338	83	2·3574033	2·9876226	2·2997740
31	4·0814954	5·3806167	3·7047675	84	2·2622625	2·8716787	2·2075378
32	4·0640806	5·3582396	3·6941206	85	2·1606068	2·7491646	2·1086753
33	4·0465466	5·3355985	3·6833581	86	2·0519543	2·6196303	2·0024741
34	4·0288902	5·3126825	3·6724787	87	1·9357942	2·4825975	1·8888448
35	4·0111083	5·2894796	3·6615142	88	1·8115858	2·3375587	1·7666121
36	3·9931977	5·2659773	3·6503966	89	1·6787574	2·1839761	1·6365019
37	3·9751549	5·2421618	3·6391898	90	1·5367053	2·0212799	1·4961503
38	3·9569765	5·2180187	3·6278924	91	1·3847931	1·8488681	1·3459302
39	3·9386592	5·1935322	3·6164698	92	1·2223502	1·6661041	1·1853551
40	3·9201994	5·1686851	3·6049196	93	1·0486713	1·4723160	1·0141803
41	3·9015934	5·1434589	3·5932721	94	0·8630150	1·2667960	0·8279118
42	3·8828376	5·1178337	3·5814924	95	0·6646031	1·0487991	0·6350412
43	3·8639282	5·0917874	3·5696106	96	0·4526193	0·8175389	0·4178535
44	3·8448614	5·0652961	3·5576242	97	0·2262087	0·5721976	0·2024611
45	3·8256334	5·0383335	3·5455308	98	$\bar{1}$·9844760	0·3119021	$\bar{1}$·9426528
46	3·8062401	5·0108708	3·5333277	99	$\bar{1}$·7264854	0·0357498	$\bar{1}$·6999244
47	3·7866776	4·9828761	3·5210123	100	$\bar{1}$·4512588	$\bar{1}$·7428037	$\bar{1}$·4258601
48	3·7669416	4·9543141	3·5085816	101	$\bar{1}$·1577756	$\bar{1}$·4320067	$\bar{1}$·1335389
49	3·7470282	4·9251456	3·4960325	102	$\bar{2}$·8449709	$\bar{1}$·1024337	$\bar{2}$·8221681
50	3·7269329	4·8953268	3·4833941	103	$\bar{2}$·5117350	$\bar{2}$·7528164	$\bar{2}$·4899585
51	3·7066516	4·8648085	3·4706640	104	$\bar{2}$·1569122	$\bar{2}$·3820170	$\bar{2}$·1367206
52	3·6861798	4·8335355	3·4578394	105	$\bar{3}$·7793001	$\bar{3}$·9867717	$\bar{3}$·7634280

LOGARITHMS OF D, N, AND M.

ENGLISH, 3 PER CENT., FEMALES.

Age.	λ D	λ N	λ M	Age.	λ D	λ N	λ M
0	4·6877587	5·9872706	4·3105061	53	3·6654605	4·8186381	3·4330192
1	4·6131539	5·9649141	4·1513199	54	3·6455266	4·7869910	3·4212012
2	4·5732644	5·9451505	4·0705479	55	3·6253990	4·7544455	3·4092781
3	4·5455206	5·9263014	4·0227385	56	3·6043151	4·7209186	3·3960158
4	4·5223902	5·9078427	3·9885040	57	3·5825868	4·6863774	3·3820056
5	4·5018484	5·8895858	3·9623840	58	3·5601455	4·6507585	3·3671182
6	4·4830352	5·8714267	3·9418780	59	3·5369212	4·6139943	3·3512809
7	4·4653425	5·8532955	3·9253388	60	3·5128417	4·5760137	3·3344453
8	4·4485011	5·8351456	3·9116908	61	3·4878313	4·5367401	3·3165196
9	4·4321721	5·8169397	3·8999071	62	3·4618103	4·4960919	3·2973983
10	4·4162948	5·7986533	3·8897799	63	3·4346936	4·4539816	3·2769950
11	4·4008083	5·7802631	3·8810799	64	3·4063899	4·4103152	3·2552430
12	4·3856606	5·7617467	3·8736057	65	3·3768003	4·3649919	3·2320604
13	4·3706519	5·7430830	3·8667068	66	3·3458180	4·3179023	3·2072705
14	4·3553131	5·7242583	3·8589496	67	3·3133269	4·2689292	3·1808203
15	4·3396097	5·7052783	3·8501437	68	3·2792003	4·2179465	3·1529695
16	4·3234857	5·6861514	3·8402094	69	3·2433005	4·1648173	3·1223189
17	4·3072936	5·6668889	3·8302256	70	3·2054774	4·1093951	3·0900247
18	4·2910321	5·6474865	3·8201921	71	3·1655678	4·0515214	4·0554425
19	4·2747000	5·6279401	3·8100704	72	3·1233939	3·9910251	3·0184137
20	4·2582960	5·6082451	3·7998985	73	3·0787629	3·9277228	2·9787436
21	4·2418186	5·5883969	3·7896757	74	3·0314656	3·8614163	2·9362487
22	4·2252664	5·5683905	3·7793644	75	2·9812756	3·7918928	2·8906480
23	4·2086380	5·5482205	3·7690385	76	2·9279480	3·7189234	2·8417925
24	4·1919318	5·5278814	3·7586242	77	2·8712189	3·6422622	2·7893127
25	4·1751464	5·5073674	3·7481203	78	2·8108040	3·5616452	2·7329125
26	4·1582802	5·4866721	3·7375995	79	2·7463977	3·4767898	2·6724407
27	4·1413315	5·4657891	3·7270255	80	2·6776722	3·3738923	2·6075163
28	4·1242986	5·4447114	3·7163613	81	2·6042762	3·2931288	2·5376161
29	4·1071799	5·4234316	3·7056421	82	2·5258344	3·1936521	2·4625401
30	4·0899737	5·4019418	3·6948673	83	2·4419462	3·0885920	2·3818395
31	4·0726781	5·3802335	3·6840363	84	2·3521844	2·9775531	2·2951619
32	4·0552913	5·3582979	3·6731486	85	2·2560948	2·8601138	2·2019840
33	4·0378115	5·3361253	3·6622032	86	2·1531950	2·7358262	2·1020013
34	4·0202367	5·3137054	3·6511996	87	2·0429730	2·6042130	1·9941415
35	4·0025651	5·2910274	3·6401370	88	1·9248868	2·4647681	1·8784535
36	3·9847947	5·2680794	3·6290495	89	1·7983630	2·3169540	1·7548175
37	3·9669234	5·2448486	3·6178671	90	1·6627959	2·1602016	1·6214557
38	3·9489491	5·2213214	3·6066578	91	1·5175467	1·9939082	1·4783024
39	3·9308698	5·1974831	3·5953865	92	1·3619420	1·8174365	1·3246404
40	3·9126834	5·1733175	3·5840523	93	1·1952734	1·6301145	1·1602284
41	3·8943876	5·1488072	3·5726886	94	1·0167961	1·4312319	0·9829899
42	3·8759802	5·1239335	3·5612606	95	0·8257281	1·2200427	0·7950454
43	3·8574590	5·0986755	3·5498012	96	0·6212489	0·9957580	0·5905632
44	3·8388216	5·0730107	3·5382755	97	0·4024990	0·7575554	0·3729672
45	3·8200658	5·0469145	3·5267519	98	0·1685785	0·5045660	0·1528691
46	3·8011892	5·0203595	3·5151268	99	$\bar{1}$·9185462	0·2358818	$\bar{1}$·8914259
47	3·7821893	4·9933158	3·5035006	100	$\bar{1}$·6514187	$\bar{1}$·9505596	$\bar{1}$·6254154
48	3·7630636	4·9657502	3·4918395	101	$\bar{1}$·3661691	$\bar{1}$·3412366	$\bar{1}$·3412366
49	3·7438097	4·9376257	3·4801428	102	$\bar{1}$·0617266	$\bar{1}$·0378248	$\bar{1}$·0378248
50	3·7244251	4·9089013	3·4684098	103	$\bar{2}$·7369747	$\bar{2}$·7143298	$\bar{2}$·7143298
51	3·7049071	4·8795308	3·4566399	104	$\bar{2}$·3907509	$\bar{2}$·3692159	$\bar{2}$·3692159
52	3·6852531	4·8494626	3·4448320	105	$\bar{2}$·0218453	$\bar{2}$·0000000	$\bar{2}$·0000000

TABLE XVII. LOGARITHMS OF D, N, AND M.

ENGLISH, 4 PER CENT., MALES.

Age.	λ D	λ N	λ M	Age.	λ D	λ N	λ M
0	4·7098992	5·9152759	4·2929028	53	3·4431190	4·5409412	3·1576287
1	4·6174799	5·8873308	4·0709033	54	3·4180567	4·5048083	3·1426192
2	4·5721900	5·8633498	3·9667208	55	3·3927904	4·4677013	3·1275685
3	4·5401633	5·8405476	3·9054946	56	3·3665092	4·4295224	3·1110568
4	4·5123849	5·8182372	3·8590742	57	3·3392248	4·3902338	3·0931436
5	4·4875254	5·7962131	3·8241808	58	3·3111550	4·3497912	3·0742911
6	4·4641600	5·7743362	3·7953582	59	3·2822211	4·3081219	3·0543738
7	4·4420019	5·7525357	3·7718967	60	3·2523423	4·2651491	3·0332897
8	4·4208167	5·7307538	3·7529231	61	3·2214349	4·2207916	3·0109978
9	4·4004307	5·7089412	3·7376754	62	3·1894115	4·1749626	2·9873674
10	4·3806786	5·6870550	3·7255377	63	3·1561794	4·1275696	2·9622857
11	4·3614477	5·6650582	3·7159320	64	3·1216404	4·0785138	2·9356990
12	4·3422167	5·6429172	3·7064910	65	3·0856890	4·0276889	2·9074954
13	4·3229858	5·6206234	3·6972678	66	3·0482119	3·9749815	2·8775383
14	4·3032338	5·5981677	3·6861492	67	3·0090869	3·9202694	2·8457493
15	4·2832517	5·5755681	3·6742648	68	2·9681818	3·8634212	2·8119361
16	4·2631807	5·5528282	3·6622702	69	2·9253537	3·8042961	2·7759586
17	4·2430312	5·5299445	3·6502163	70	2·8804474	3·7427422	2·7376938
18	4·2228010	5·5069122	3·6380551	71	2·8332951	3·6785966	2·6969894
19	4·2024879	5 4837268	3·6257877	72	2·7837148	3·6116836	2·6536589
20	4·1820897	5·4603834	3·6134630	73	2·7315097	3·5418152	2·6074136
21	4·1616043	5·4368767	3·6010342	74	2·6764672	3·4687886	2·5581557
22	4·1410294	5·4132014	3·5885495	75	2·6183576	3·3923867	2·5056492
23	4·1203626	5·3893518	3·5759162	76	2·5569333	3·3123763	2·4495427
24	4·0996015	5·3653218	3·5632731	77	2·4919277	3·2285075	2·3897525
25	4·0787437	5·3411049	3·5504839	78	2·4230544	3·1405122	2·3257888
26	4·0577865	5·3166948	3·5375938	79	2·3500061	3·0481037	2·2574777
27	4·0367273	5·2920841	3·5246480	80	2·2724533	2·9509753	2·1843791
28	4·0155635	5·2672655	3·5116023	81	2·1900440	2·8487991	2·1063113
29	3·9942922	5·2422310	3·4984568	82	2·1024018	2·7412253	2·0228135
30	3·9729106	5·2169721	3·4852550	83	2·0091258	2·6278805	1·9334903
31	3·9514159	5·1914800	3·4719104	84	1·9097889	2·5083672	1·8379210
32	3·9298050	5·1657450	3·4585081	85	1·8039370	2·3822618	1·7356859
33	3·9080748	5·1397569	3·4450056	86	1·6910884	2·2491139	1·6260651
34	3·8862223	5·1135048	3·4314022	87	1·5707322	2·1084456	1·5089833
35	3·8642443	5·0869773	3·4177394	88	1·4423277	1·9597495	1·3832479
36	3·8421375	5·0591616	3·4039333	89	1·3053032	1·8024878	1·2496214
37	3·8198986	5·0330441	3·3900654	90	1·1590550	1·6360905	1·1056975
38	3·7975242	5·0056104	3·3761352	91	1·0029466	1·4599561	0·9518716
39	3·7750107	4·9778448	3·3621015	92	0·8363076	1·2734481	0·7876588
40	3·7523548	4·9497303	3·3479631	93	0·6584326	1·0758972	0·6128368
41	3·7295527	4·9212482	3·3337587	94	0·4685802	0·8665945	0·4228196
42	3·7066008	4·8923784	3·3194478	95	0·2659723	0·6447930	0·2262905
43	3·6834953	4·8630988	3·3050685	96	0·0497923	0·4097134	0·0052234
44	3·6602324	4·8333854	3·2906198	97	$\bar{1}$·8191855	0·1605286	$\bar{1}$·7861833
45	3·6368082	4·8032115	3·2761009	98	$\bar{1}$·5732568	$\bar{1}$·8963608	$\bar{1}$·5221833
46	3·6132188	4·7725480	3·2615106	99	$\bar{1}$·3110700	$\bar{1}$·6163705	$\bar{1}$·2757719
47	3·5894601	4·7413623	3·2468478	100	$\bar{1}$·0316474	$\bar{1}$·3195224	$\bar{2}$·9978231
48	3·5655281	4·7096187	3·2321113	101	$\bar{2}$·7339680	$\bar{1}$·0047512	$\bar{2}$·7015680
49	3·5414185	4·6772770	3·2172998	102	$\bar{2}$·4169672	$\bar{2}$·6711728	$\bar{2}$·3856063
50	3·5171272	4·6442920	3·2024501	103	$\bar{2}$·0795352	$\bar{2}$·3180633	$\bar{2}$·0492180
51	3·4926497	4·6106137	3·1875617	104	$\bar{3}$·7205163	$\bar{3}$·9444827	$\bar{3}$·6901961
52	3·4679818	4·5761850	3·1726340	105	$\bar{3}$·3387081	$\bar{3}$·5440680	$\bar{3}$·3010300

TABLE XVIII. LOGARITHMS OF D, N, AND M.

ENGLISH, 4 PER CENT., FEMALES.

Age.	λD	λN	λM	Age.	λD	λN	λM
0	4·6877587	5·9082009	4·2453304	53	3·4430664	4·5567470	3·1422819
1	4·6089578	5·8812388	4·0561467	54	3·4189364	4·5219630	3·1285527
2	4·5648721	5·8573952	3·9552824	55	3·3946127	4·4862786	3·1147508
3	4·5329323	5·8346665	3·8937795	56	3·3693327	4·4496074	3·0995302
4	4·5056057	5·8124270	3·8489185	57	3·3434083	4·4119182	3·0835070
5	4·4808678	5·7904534	3·8142609	58	3·3167709	4·3731464	3·0665650
6	4·4578585	5·7686227	3·7868289	59	3·2893505	4·3332239	3·0486298
7	4·4359697	5·7468526	3·7645948	60	3·2610748	4·2920786	3·0296551
8	4·4149322	5·7250885	2·7462041	61	3·2318683	4·2496331	3·0095462
9	4·3944071	5·7032873	3 7303170	62	3·2016512	4·2058054	2·9881943
10	4·3743336	5·6814202	3·7166752	63	3·1703384	4·1605070	2·9655123
11	4·3546511	5·6594602	3·7049814	64	3·1378385	4·1136434	2·9414366
12	4·3353072	5·6373817	3·6949682	65	3·1040529	4·0651125	2·9158864
13	4·3161024	5·6151603	3·6857632	66	3·0688745	4·0148044	2·8886778
14	4·2965675	5·5927798	3·6754555	67	3·0321872	3·9626016	2·8597645
15	4·2766680	5·5702484	3·6637939	68	2·9938645	3·9083766	2·8294408
16	4·2563478	5·5475763	3·6506740	69	2·9537686	3·8519921	2·7961967
17	4·2359596	5·5247780	3·6375193	70	2·9117494	3·7933005	2·7612983
18	4·2155020	5·5018492	3·6243308	71	2·8676436	3·7321425	2·7240632
19	4·1949738	5·4787862	3·6110584	72	2·8212736	3·6683465	2·6843310
20	4·1743737	5·4555844	3·5977520	73	2·7724465	3·6017280	2·6419097
21	4·1537001	5·4322393	3·5844122	74	2·7209531	3·5320882	2·5966164
22	4·1329519	5·4087463	3·5709904	75	2·6665670	3·4592134	2·5481662
23	4·1121273	5·3851003	3·5575839	76	2·6090433	3·3828741	2·4964163
24	4·0912251	5·3612957	3·5440969	77	2·5481181	3.3028236	2·4409909
25	4·0702436	5·3373272	3·5305294	78	2·4835071	3·2187974	2·3815930
26	4·0491812	5·3131885	3·5169761	79	2·4149047	3·1305119	2·3180798
27	4·0280364	5·2888732	3·5033909	80	2·3419830	3·0376636	2·2500681
28	4·0068074	5·2643747	3·4897272	81	2·2643909	2·9399272	2·1770267
29	3·9854926	5·2396858	3·4760314	82	2·1817531	2·8369556	2·0987635
30	3·9640902	5·2147987	3·4623033	83	2·0936686	2·7283781	2·0148287
31	3·9425985	5·1897051	3·4485437	84	1·9997107	2·6137990	1·9248734
32	3·9210156	5·1643963	3·4347525	85	1·8994251	2·4927968	1·8283724
33	3·8993397	5·1388630	3·4209300	86	1·7923291	2·3649231	1·7250283
34	3·8775688	5·1130950	3·4070765	87	1·6779110	2·2297005	1·6137509
35	3·8557011	5·0870814	3·3931923	88	1·5556287	2·0866232	1·4946068
36	3·8337346	5·0608105	3·3793216	89	1·4249088	1·9351534	1·3674938
37	3·8116671	5·0342700	3·3653777	90	1·2851456	1·7747220	1·2306022
38	3·7894968	5·0074459	3·3514470	91	1·1357002	1·6047270	1·0838823
39	3·7672214	4·9803237	3·3374874	92	0·9758995	1·4245305	0·9266151
40	3·7448388	4·9528873	3·3234990	93	0·8050348	1·2334606	0·7585713
41	3·7223469	4·9251191	3·3095248	94	0·6223613	1·0308101	0·5776411
42	3·6997434	4·8970003	3·2955231	95	0·4270972	0·8158300	0·3860171
43	3·6770261	4·8685103	3·2815363	96	0·2184219	0·5877334	0·1777673
44	3·6541926	4·8396260	3·2675235	97	$\bar{1}$·9954759	0·3456873	$\bar{1}$·9563606
45	3·6312407	4·8103223	3·2535680	98	$\bar{1}$·7573593	0·0888446	$\bar{1}$·7326350
46	3·6081679	4·7805720	3·2395479	99	$\bar{1}$·5031309	$\bar{1}$·8162413	$\bar{1}$·4671640
47	3·5849719	4·7503442	3·2255855	100	$\bar{1}$·2318072	$\bar{1}$·5269851	$\bar{1}$·1972806
48	3·5616501	4·7196051	3·2116418	101	$\bar{2}$·9423616	$\bar{1}$·2201081	$\bar{2}$·9090209
49	3·5382001	4·6883170	3·1977180	102	$\bar{2}$·6337229	$\bar{2}$·8943161	$\bar{2}$·6009729
50	3·5146193	4·6564375	3·1838150	103	$\bar{2}$·3047749	$\bar{2}$·5490033	$\bar{2}$·2718416
51	3·4909052	4·6239196	3·1699340	104	$\bar{3}$·9543549	$\bar{2}$·1818436	$\bar{3}$·9190781
52	3·4670551	4·5907096	3·1560762	105	$\bar{3}$·5812533	$\bar{3}$·7923917	$\bar{3}$·5440680

TABLE XIX. ANNUITIES ON SINGLE LIVES.
ENGLISH, 3 AND 4 PER CENT., MALES.

Age.	Present value of annuity of £1.		Life annuity for £100. 3 per Cent.
	3 per Cent.	4 per Cent.	
0	18·2167	15·0464	5·4895
1	21·3199	17·6145	4·6904
2	22·4358	18·5506	4·4572
3	22·9206	18·9703	4·3629
4	23·1996	19·2233	4·3104
5	23·3301	19·3558	4·2863
6	23·3829	19·4257	4·2766
7	23·3703	19·4425	4·2789
8	23·3026	19·4144	4·2914
9	23·1877	19·3475	4·3126
10	23·0333	19·2477	4·3415
11	22·8447	19·1192	4·3774
12	22·6494	18·9848	4·4151
13	22·4472	18·8444	4·4549
14	22·2658	18·7212	4·4912
15	22·0901	18·6027	4·5269
16	21·9125	18·4826	4·5636
17	21·7324	18·3604	4·6014
18	21·5497	18·2358	4·6404
19	21·3645	18·1090	4·6807
20	21·1765	17·9799	4·7222
21	20·9859	17·8483	4·7651
22	20·7925	17·7142	4·8094
23	20·5962	17·5776	4·8553
24	20·3969	17·4383	4·9027
25	20·1947	17·2962	4·9518
26	19·9893	17·1513	5·0027
27	19·7807	17·0035	5·0554
28	19·5688	16·8526	5·1102
29	19·3536	16·6986	5·1670
30	19·1347	16·5413	5·2261
31	18·9123	16·3806	5·2876
32	18·6861	16·2163	5·3516
33	18·4559	16·0483	5·4183
34	18·2217	15·8765	5·4880
35	17·9833	15·6645	5·5607
36	17·7404	15·5205	5·6368
37	17·4929	15·3360	5·7166
38	17·2407	15·1468	5·8002
39	16·9835	14 9527	5·8881
40	16·7209	14·7534	5·9805
41	16·4528	14·5488	6·0780
42	16·1789	14·3383	6·1809
43	15·8989	14·1218	6·2897
44	15·6125	13·8989	6·4051
45	15·3192	13·6691	6·5277
46	15·0188	13·4321	6·6583
47	14·7108	12·1874	6·7977
48	14·3947	12·9345	6·9470
49	14·0701	12·6728	7·1072
50	13·7365	12·4018	7·2799
51	13·3932	12·1209	7·4665
52	13·0396	11·8293	7·6689
53	12·6751	11·5263	7·8895
54	12·2991	11·2110	8·1307
55	11·9104	10·8826	8·3960
56	11·5318	10·5615	8·6717
57	11·1614	10·2463	8·9595
58	10·7921	9·9304	9·2660
59	10·4247	9·6145	9·5926
60	10·0598	9·2993	9·9406
61	9·6979	8·9852	10·3115
62	9·3397	8·6728	10·7070
63	8·9854	8·3625	11·1291
64	8·6357	8·0547	11·5799
65	8·2908	7·7498	12·0616
66	7·9511	7·4483	12·5769
67	7·6170	7·1505	13·1285
68	7·2888	6·8567	13·7196
69	6·9669	6·5673	14·3536
70	6·6516	6·2827	15·0341
71	6·3431	6·0033	15·7652
72	6·0418	5·7293	16·5513
73	5·7480	5·4611	17·3972
74	5·4620	5·1990	18·3084
75	5·1839	5·9433	19·2904
76	4·9141	4·6943	20·3497
77	4·6526	4·4523	21·4932
78	4·3998	4·2174	22·7284
79	4·1557	3·9900	24·0636
80	3·9203	3·7700	25·5079
81	3·6939	3·5578	27·0713
82	3·4765	3·3533	28·7647
83	3·2680	3·1568	30·6002
84	3·0683	2·9681	32·5910
85	2 8776	2·7873	34·7517
86	2·6955	2·6143	37·0985
87	2·5221	2·4492	39·6493
88	2·3572	2·2917	42·4238
89	2·2005	2·1418	45·4441
90	2·0519	1·9994	48·7346
91	1·9112	1·8642	52·3226

TABLE XX. ANNUITIES ON SINGLE LIVES.

ENGLISH, 3 AND 4 PER CENT., FEMALES.

Age.	**Present value of annuity of £1.**		**Life annuity for £100.**
	3 per Cent.	**4 per Cent.**	**3 per Cent.**
0	18·9302	15·6128	5·2826
1	21·4781	17·7189	4·6559
2	22·5443	18·6121	4·4357
3	23·0315	19·0325	4·3419
4	23·2914	19·2685	4·2934
5	23·4195	19·3979	4·2699
6	23·4563	19·4533	4·2632
7	23·4317	19·4589	4·2677
8	23·3582	19·4247	4·2812
9	23·2531	19·3648	4·3005
10	23·1190	19·2809	4·3255
11	22·9582	19·1748	4 3557
12	22·7731	19·0482	4·3911
13	22·5739	18·9094	4·4299
14	22·3854	18·7794	4·4672
15	22·2096	18·6598	4·5025
16	22·0497	18·5537	4·5352
17	21·8873	18·4455	4·5689
18	21·7224	18·3351	4·6035
19	21·5549	18.2226	4·6393
20	21·3846	18·1078	4·6763
21	21·2115	17·9906	4·7144
22	21·0356	17·8710	4·7539
23	20·8566	17·7488	4·7946
24	20·6745	17·6239	4·8369
25	20·4892	17·4962	4·8806
26	20·3006	17·3657	4·9260
27	20·1085	17·2321	4·9730
28	19·9128	17·0954	5·0219
29	19.7134	16·9553	5·0727
30	19·5101	16·8118	5·1255
31	19·3028	16·6647	5·1806
32	19·0912	16·5138	5·2380
33	18·8753	16·3589	5·2979
34	18·6548	16·1999	5·3605
35	18·4295	16·0365	5·4261
36	18·1993	15·8685	5·4947
37	17·9638	15·6956	5·5668
38	17·7229	15·5177	5·6424
39	17·4762	15·3344	5·7221
40	17·2236	15·1454	5·8060
41	16·9647	14·9504	5·8946
42	16·6992	14·7491	5·9883
43	16·4268	14·5412	6·0876
44	16·1470	14·3262	6·1931
45	15·8597	14·1036	6·3053
46	15·5642	13·8732	6·4250
47	15·2602	13·6343	6·5530
48	14·9473	13·3865	6·6902
49	14·6249	13·1292	6·8377
50	14·2924	12·8618	6·9967
51	13·9494	12·5836	7·1688
52	13·5952	12·2940	7·3555
53	13·2291	11·9921	7·5591
54	12·8505	11·6773	7·7818
55	12·4600	11·3500	8·0257
56	12·0799	11·0303	8·2782
57	11·6996	10·7087	8·5473
58	11·3201	10·3861	8·8339
59	10·9419	10·0630	9·1392
60	10·5657	9·7400	9·4646
61	10·1920	9·4175	9·8116
62	9·8214	9·0961	10·1819
63	9·4541	8·7762	10·5774
64	9·0908	8·4581	11·0001
65	8·7318	8·1424	11·4524
66	8·3774	7 8294	11·9368
67	8·0282	7·5195	12·4561
68	7·6845	7·2132	13·0132
69	7·3467	6·9109	13·6115
70	7·0153	6·6129	14·2546
71	6·6905	6·3198	14·9466
72	6·3728	6·0319	15·6917
73	6·0625	5·7497	16·4948
74	5·7601	5·4734	17·3609
75	5·4657	5·2036	18·2958
76	5·1798	4·9406	19·3057
77	4·9026	4·6847	20·3973
78	4·6343	4·4361	21·5782
79	4.3752	4·1953	22·8563
80	4·1253	3·9623	24·2406
81	3·8849	3·7374	25·7409
82	3·6539	3·5207	27·3680
83	3·4325	3·3123	29·1335
84	3·2205	3·1123	31·0506
85	3·0181	2·9208	33·1336
86	2·8250	2·7376	35·3983
87	2·6412	2 7628	37·8621
88	2·4664	2·3962	40·5446
89	2·3006	2·2378	43·4672
99	2·1434	2·0873	46·6539
91	1·9948	1·9446	50·1314

TABLE XXI. VALUE OF POLICIES—AFTER BEING IN FORCE FROM 1 TO 10 YEARS, ENGLISH, 3 PER CENT.

Age at entry	1	2	3	4	5	6	7	8	9	10	Age at entry
20	·00860	·01733	·02615	·03517	·04428	·05352	·06293	·07249	·08219	·09208	20
21	·00879	·01769	·02678	·03598	·04530	·05479	·06443	·07422	·08419	·09428	21
22	·00899	·01816	·02744	·03684	·04642	·05614	·06602	·07608	·08627	·09666	22
23	·00924	·01860	·02809	·03775	·04756	·05753	·06768	·07796	·08845	·09910	23
24	·00937	·01904	·02879	·03869	·04875	·05900	·06937	·07996	·09071	·10166	24
25	·00968	·01952	·02952	·03968	·05002	·06049	·07118	·08203	·09308	·10435	25
26	·00993	·02003	·03028	·04072	·05230	·06209	·07305	·08421	·09558	·10713	26
27	·01019	·02055	·03110	·04178	·05268	·06375	·07502	·08651	·09817	·11007	27
28	·01045	·02111	·03191	·04292	·05411	·06549	·07710	·08888	·10091	·11319	28
29	·01076	·02167	·03279	·04410	·05560	·06733	·07924	·09139	·10381	·11645	29
30	·01103	·02228	·03371	·04534	·05720	·06924	·08152	·09407	·10685	·11989	30
31	·01136	·02292	·03468	·04667	·05884	·07126	·08395	·09688	·11006	·12353	31
32	·01169	·02358	·03571	·04802	·06059	·07342	·08650	·09983	·11346	·12735	32
33	·01204	·02430	·03676	·04948	·06247	·07569	·08918	·10297	·11703	·13142	33
34	·01241	·02503	·03789	·05104	·06453	·07808	·09204	·10627	·12083	·13573	34
35	·01278	·02581	·03912	·05268	·06650	·08064	·09505	·10979	·12488	·14033	35
36	·01318	·02667	·04040	·05440	·06872	·08332	·09825	·11354	·12919	·14523	36
37	·01365	·02756	·04175	·05626	·07105	·08619	·10167	·11754	·13380	·15046	37
38	·01412	·02850	·04321	·05821	·07355	·08925	·10534	·12182	·13871	·15603	38
39	·01460	·02952	·04473	·06030	·07622	·09254	·10925	·12639	·14396	·16199	39
40	·01515	·03058	·04638	·06254	·07910	·09606	·11345	·13128	·14957	·16840	40
41	·01552	·03156	·04797	·06479	·08201	·09967	·11778	·13636	·15548	·17516	41
42	·01629	·03296	·05004	·06754	·08548	·10387	·12274	·14216	·16216	·18276	42
43	·01694	·03430	·05209	·07032	·08902	·10821	·12795	·14828	·16922	·19077	43
44	·01768	·03577	·05432	·07334	·09285	·11294	·13361	·15482	·17683	·19947	44
45	·01841	·03730	·05666	·07653	·09698	·11802	·13971	·16202	·18507	·20888	45
46	·01924	·03897	·05921	·08004	·10148	·12358	·14631	·16979	·19404	·21768	46
47	·02014	·04077	·06200	·08387	·10639	·12957	·15351	·17824	·20234	·22591	47
48	·02107	·04275	·06506	·08805	·11170	·13613	·16137	·18596	·21001	·23403	48
49	·02213	·04492	·06840	·09256	·11752	·14330	·16843	·19300	·21753	·24192	49
50	·02330	·04731	·07202	·09754	·12391	·14960	·17473	·19981	·22475	·24950	50

Age at entry	1	2	3	4	5	6	7	8	9	10	Age at entry
51	·02459	·04988	·07602	·10301	·12932	·15504	·18073	·20626	·23159	·25676	51
52	·02596	·05275	·08042	·10739	·13377	·16009	·18627	·21224	·23804	·26356	52
53	·02750	·05591	·08360	·11068	·13771	·16458	·19125	·21773	·24393	·26980	53
54	·02922	·05768	·08553	·11332	·14095	·16837	·19561	·22255	·24915	·27543	54
55	·02933	·05800	·08663	·11510	·14334	·17139	·19915	·22655	·25362	·28034	55
56	·02954	·05903	·08836	·11746	·14636	·17495	·20318	·23107	·25859	·28570	56
57	·03038	·06060	·09058	·12036	·14982	·17891	·20765	·23602	·26395	·29146	57
58	·03118	·06210	·09281	·12320	·15320	·18284	·21209	·24090	·26927	·29710	58
59	·03195	·06364	·09500	·12597	·15656	·18676	·21649	·24577	·27450	·30265	59
60	·03274	·06513	·09713	·12873	·15992	·19063	·22088	·25056	·27964	·30812	60
61	·03352	·06659	·09927	·13151	·16326	·19453	·22521	·25528	·28472	·31358	61
62	·03425	·06806	·10141	·13427	·16662	·19837	·22947	·25593	·28979	·31898	62
63	·03500	·06954	·10356	·13706	·16992	·20213	·23369	·26459	·29483	·32417	63
64	·03577	·07102	·10574	·13979	·17317	·20587	·23790	·26923	·29964	·32933	64
65	·03653	·07253	·10786	·14248	·17640	·20961	·24210	·27364	·30443	·33442	65
66	·03734	·07400	·10993	·14514	·17962	·21334	·24608	·27804	·30916	·33922	66
67	·03810	·07543	·11200	·14781	·18283	·21685	·25005	·28238	·31361	·34408	67
68	·03882	·07684	·11407	·15048	·18585	·22036	·25396	·28643	·31810	·34857	68
69	·03954	·07827	·11615	·15295	·18885	·22381	·25761	·29055	·32224	·35290	69
70	·04029	·07972	·11805	·15541	·19183	·22700	·26130	·29430	·32622	·35685	70
71	·04006	·08100	·11994	·15788	·19454	·23027	·26466	·29792	·32985	·36072	71
72	·04168	·08230	·12184	·16009	·19735	·23320	·26789	·30117	·33336	·36429	72
73	·04234	·08361	·12352	·16240	·19982	·23601	·27075	·30434	·33661	·36756	73
74	·04305	·08472	·12532	·16440	·20219	·23847	·27355	·30725	·33956	·37052	74
75	·04356	·08598	·12681	·16630	·20424	·24089	·27609	·30987	·34221	·37320	75
76	·04427	·08697	·12826	·16794	·20626	·24306	·27838	·31220	·34459	·37509	76
77	·04475	·08795	·12946	·16954	·20805	·24499	·28037	·31426	·34620	·37703	77
78	·04523	·08870	·13068	·17098	·20966	·24669	·28215	·31559	·34786	·37769	78
79	·04555	·08950	·13171	·17222	·21101	·24814	·28318	·31697	·34724	·37972	79
80	·04596	·09019	·13263	·17326	·21217	·24889	·28429	·31711	·35004	·37948	80

Age.	Males.		Females.		Total.		Age.	Males.		Females.		Total.	
	l_x	d_x	l_x	d_x	l_x	d_x		l_x	d_x	l_x	d_x	l_x	d_x
0	51274	8170	48726	6461	100000	14631	52	22580	404	22531	358	45111	762
1	43104	2716	42265	2551	85369	5267	53	22176	406	22173	359	44349	765
2	40388	1370	39714	1340	80102	2710	54	21770	409	21814	363	43584	772
3	39018	954	38374	899	77392	1853	55	21361	450	21451	404	42812	854
4	38064	679	37475	659	75539	1338	56	20911	488	21047	426	41958	914
5	37385	542	36816	505	74201	1047	57	20423	512	20621	451	41044	963
6	36843	432	36311	402	73154	834	58	19911	538	20170	477	41081	1015
7	36411	346	35909	330	72320	676	59	19373	565	19693	503	39066	1068
8	36065	278	35579	285	71644	563	60	18808	591	19190	530	37998	1121
9	35787	223	35294	246	71081	469	61	18217	618	18660	558	36877	1176
10	35564	179	35048	213	70612	392	62	17599	645	18102	586	35701	1231
11	35385	179	34835	185	70220	364	63	16954	670	17516	613	34470	1283
12	35206	178	34650	173	69856	351	64	16284	694	16903	639	33187	1333
13	35028	218	34477	197	69505	415	65	15590	717	16264	666	31854	1383
14	34810	236	34280	226	69090	462	66	14873	737	15598	690	30471	1427
15	34574	241	34054	257	68628	498	67	14136	756	14908	703	29044	1459
16	34333	245	33797	260	68130	505	68	13380	772	14205	745	27585	1517
17	34088	250	33537	263	67625	513	69	12608	784	13460	752	26068	1536
18	33838	255	33274	267	67112	522	70	11824	792	12708	768	24532	1560
19	33583	259	33007	270	66590	529	71	11032	796	11940	780	22972	1576
20	33324	264	32737	273	66061	537	72	10236	797	11160	788	21396	1585
21	33060	268	32464	277	65524	545	73	9439	791	10372	791	19811	1582
22	32792	274	32187	279	64979	553	74	8648	780	9581	790	18229	1570
23	32518	277	31908	283	64426	560	75	7868	765	8791	782	16654	1547
24	32241	283	31625	287	63866	570	76	7103	742	8009	770	15112	1512
25	31958	288	31338	289	63296	577	77	6361	716	7239	752	13600	1468
26	31670	292	31049	292	62719	584	78	5645	683	6487	726	12132	1409
27	31378	297	30757	296	62135	593	79	4962	646	5761	695	10723	1341
28	31081	302	30461	299	61542	601	80	4316	603	5066	660	9382	1263
29	30779	306	30162	302	60941	608	81	3713	557	4406	618	8119	1175
30	30473	312	29860	305	60333	617	82	3156	508	3788	572	6944	1080
31	30161	316	29555	308	59716	624	83	2648	457	3216	522	5864	979
32	29845	321	29247	311	59092	632	84	2191	405	2694	470	4885	875
33	29524	326	28936	314	58460	640	85	1786	354	2224	416	4010	770
34	29198	330	28622	317	57820	647	86	1432	303	1808	364	3240	667
35	28868	336	28305	319	57173	655	87	1129	256	1444	311	2573	567
36	28532	340	27986	323	56518	663	88	873	210	1133	260	2006	470
37	28192	344	27663	325	55855	669	89	663	171	873	215	1536	386
38	27848	349	27338	328	55186	677	90	492	135	658	173	1150	308
39	27499	354	27010	331	54509	685	91	357	104	485	136	842	240
40	27145	358	26679	333	53824	691	92	253	78	349	104	602	182
41	26787	363	26346	336	53133	699	93	175	58	245	78	420	136
42	26424	367	26010	338	52434	705	94	117	40	167	56	284	96
43	26057	371	25672	341	51729	712	95	77	29	111	40	188	69
44	25686	375	25331	342	51017	717	96	48	18	71	27	119	45
45	25311	379	24989	346	50300	725	97	30	13	44	17	74	30
46	24932	383	24643	347	49575	730	98	17	7	27	12	44	19
47	24549	387	24296	349	48845	736	99	10	5	15	7	25	11
48	24162	391	23947	351	48109	742	100	5	3	9	4	14	7
49	23771	394	23596	353	47367	747	101	3	1	5	2	7	4
50	23377	397	23243	355	46620	752	102	1		2	1	4	2
51	22980	400	22888	357	45868	757	103	1		1		2	

TABLE XXIII. MORTALITY TABLE, CARLISLE.

Age.	l_x	d_x	Expectation of Life.	Proportion of Death.
0	10000	1539	38·72	·153900
1	8461	682	44·68	·080605
2	7779	505	47·55	·064918
3	7274	276	49·82	·037943
4	6998	201	50.76	·028723
5	6797	121	51·25	·017802
6	6676	82	51·17	·012283
7	6594	58	50·80	·008796
8	6536	43	50·24	·006579
9	6493	33	49·57	·005082
10	6460	29	48·82	·004489
11	6431	31	48·04	·004820
12	6400	32	47·27	·005000
13	6368	33	46·51	·005182
14	6335	35	45·75	·005525
15	6300	39	45·00	·006191
16	6261	42	44·27	·006708
17	6219	43	43·57	·006914
18	6176	43	42·87	·006962
19	6133	43	42·17	·007011
20	6090	43	41·46	·007061
21	6047	42	40·75	·006946
22	6005	42	40·04	·006994
23	5963	42	39·31	·007043
24	5921	42	38·59	·007093
25	5879	43	37·86	·007314
26	5836	43	37·14	·007368
27	5793	45	36·41	·007768
28	5748	50	35·69	·008699
29	5698	56	35·00	·009828
30	5642	57	34·34	·010103
31	5585	57	33·68	·010206
32	5528	56	33·03	·010130
33	5472	55	32·36	·010051
34	5417	55	31·68	·010153
35	5362	55	31·00	·010257
36	5307	56	30·32	·010552
37	5251	57	29·64	·010855
38	5194	58	28·96	·011167
39	5136	61	28·28	·011877
40	5075	66	27·61	·013005
41	5009	69	26·97	·013775
42	4940	71	26.34	·014373
43	4869	71	25·71	·014582
44	4798	71	25·09	·014798
45	4727	70	24·46	·014809
46	4657	69	23·82	·014816
47	4588	67	23·17	·014603
48	4521	63	22·50	·013935
49	4458	61	21·81	·013683
50	4397	59	21·11	·013418
51	4338	62	20·39	·014292
52	4276	65	19·68	·015201
53	4211	68	18·97	·016148
54	4143	70	18·28	·016896
55	4073	73	17·58	·017923
56	4000	76	16·89	·019000
57	3924	82	16·21	·020897
58	3842	93	15·55	·024206
59	3749	106	14·92	·028274
60	3643	122	14·34	·033489
61	3521	126	13·82	·035785
62	3395	127	13·31	·037408
63	3268	125	12·81	·038250
64	3143	125	12·30	·039771
65	3018	124	11·79	·041087
66	2894	123	11·27	·042502
67	2771	123	10·75	·044388
68	2648	123	10·23	·046450
69	2525	124	9·70	·049109
70	2401	124	9·18	·051645
71	2277	134	8·65	·058849
72	2143	146	8·16	·068129
73	1997	156	7·72	·078117
74	1841	166	7·33	·090168
75	1675	160	7·01	·095522
76	1515	156	6·69	·102970
77	1359	146	6·40	·107432
78	1213	132	6·12	·108821
79	1081	128	5·80	·118409
80	953	116	5·51	·121721
81	837	112	5·21	·133811
82	725	102	4·93	·140690
83	623	94	4·65	·150883
84	529	84	4·39	·158790
85	445	78	4·12	·175281
86	367	71	3·90	·193461
87	296	64	3·71	·216216
88	232	51	3·59	·219828
89	181	39	3·47	·215470
90	142	37	3·28	·260563
91	105	30	3·26	·285714
92	75	21	3·37	·280000
93	54	14	3·48	·259259
94	40	10	3·53	·250000
95	30	7	3·53	·233333
96	23	5	3·46	·217391
97	18	4	3·28	·222222
98	14	3	3·07	·214286
99	11	2	2·77	·181818
100	9	2	2·28	·222222
101	7	2	1·79	·285714
102	5	2	1·30	·400000
103	3	2	·83	·666666
104	1	1	·50	·000000

Years defd.	Age 14.	Age 15.	Age 16.	Age 17.	Age 18.	Age 19.	Years defd.
0	19·08182	18·95535	18·83635	18·72211	18·60656	18·48649	0
1	18·12560	17·99976	17·88126	17·76722	17·65172	17·53170	1
2	17·21184	17·08709	16·96925	16·85545	16·74003	16·62610	2
3	16·33912	16·21559	16·09843	15·98489	15·86961	15·74966	3
4	15·50577	15·38345	15·26697	15 15373	15·03847	14·91855	4
5	14·71005	14·58892	14·47314	14·30008	14·24489	14·12504	5
6	13·95030	13·83034	13·71514	13·69230	13·48721	13·36745	6
7	13·22493	13·10601	12·99139	12·87880	12·76383	12·64434	7
8	12·53230	12·41440	12·30038	12·18805	12·07337	11·95415	8
9	11·87097	11·75408	11·64066	11·52874	11·41435	11·29567	9
10	11·23956	11·12366	11·01095	10·89945	10·78560	10·66802	10
11	10·63674	10·52192	10·40992	10·29906	10·18630	10·07045	11
12	10·06134	9·94759	9·83651	9·72679	9·61571	9·50166	12
13	9·51215	9·39964	9·28994	9·18194	9·07260	8·96033	13
14	8·98818	8·87734	8·76955	8·66334	8·55572	8·44509	14
15	8·48875	8·38007	8·27424	8·16977	8·06375	7·95465	15
16	8·01325	7·90676	7·80284	7 69999	7·59545	7·48787	16
17	7·56065	7·45630	7·35416	7·25282	7·14974	7·04363	17
18	7·12990	7·02754	6·92708	6·82722	6·72557	6·62100	18
19	6·71992	6·61943	6·52059	6·42218	6·32202	6·21902	19
20	6·32967	6·23099	6·13374	6·03683	5·93820	5·83683	20
21	5·95824	5·86133	5·76570	5·67032	5·57326	5·47370	21
22	5·60475	5·50963	5·41565	5·32185	5·22653	5·12907	22
23	5·26845	5·17513	5·08283	4·99076	4·89747	4·80227	23
24	4·94859	4·85709	4·76660	4·67654	4·58542	4·49255	24
25	4·64447	4·55491	4·46650	4·37857	4·28969	4·19909	24
26	4·35553	4·26813	4·18191	4·09618	4·00947	3·92109	26
27	4·08130	3·99619	3·91120	3·82861	3·74403	3·65774	27
28	3·82126	3·73846	3·65665	3·57513	3·49257	3·40827	28
29	3·57481	3·49425	3·41456	3·33502	3·25437	3·17190	29
30	3·34129	3·26291	3·18523	3·10756	3·02867	2·94778	30
31	3·12008	3·04377	2·96798	2·89204	2·81467	2·73524	31
32	2·91053	2·83617	2·76215	2·68770	2·61173	2·53361	32
33	2·71202	2·63948	2·56698	2·49391	2 41920	2·34251	33
34	2·52394	2·45298	2·38190	2·31007	2·23673	2·16155	34
35	2·34561	2·27612	2·20631	2·13583	2·06395	1·99036	35
36	2·17648	2·10833	2·03990	1 97084	1·90049	1·82854	36
37	2·01604	1·94931	1·88232	1·81476	1·74597	1·67573	37
38	1·86398	1·79873	1·73324	1·66721	1·60006	1·53159	38
39	1·71999	1·65627	1·59232	1·52788	1·46243	1·39589	39
40	1·58377	1·52161	1·45925	1·39646	1·33285	1·26856	40
41	1·45500	1·39445	1·33373	1·27273	1·21128	1·14960	41
42	1·33341	1·27450	1·21556	1·15664	1·09769	1·03904	42
43	1·21871	1·16158	1·10468	1·048817	0·99212	0·93654	43
44	1·11073	1·05563	1·00109	0·94737	0·89425	0·84167	44
45	1·00942	0·95663	0·90481	0·85391	0·80366	0·75393	45
46	0·91476	0·86463	0·81555	0·76741	0·71989	0·67293	46
47	0·82679	0·77934	0·73293	0·68741	0·64254	0·59824	47
48	0·74522	0·70039	0·65653	0 61356	0·57122	0·52947	48
49	0·66973	0·62738	0·58599	0·54546	0·50556	0·46629	49
50	0·59992	0·55997	0·52095	0·48276	0·44523	0·40835	50
51	0·53546	0·49782	0·46107	0·42515	0·38991	0·35539	51
52	0·47603	0·55997	0·40604	0·37233	0·33934	0·30708	52
53	0·42131	0·49782	0·35559	0·32403	0·29322	0·26337	53
54	0·37103	0·44060	0·30947	0·27999	0·25148	0·22421	54

Years defer'd	Age 14.	Age 15.	Age 16.	Age 17.	Age 18.	Age 19.	Years defer'd
55	0·32493	0·38802	0·26741	0·24014	0·21408	0·18949	55
56	0·28279	0·33981	0·22934	0·20443	0·18093	0·15912	56
57	0·24435	0·29573	0·19524	0·17277	0·15193	0·13271	57
58	0·20977	0·25554	0·16500	0·14508	0·12671	0·10992	58
59	0·17841	0·21917	0·13855	0·12100	0·10496	0·09037	59
60	0·15078	0·18658	0·11555	0·10022	0·08629	0·07361	60
61	0·12661	0·15768	0·09571	0·08240	0·07029	6·05941	61
62	0·10560	0·13241	0·07868	0·06712	0·05673	0·04742	62
63	0·08747	0·11043	0·06409	0·05417	0·04527	0·03743	63
64	0·07191	0·09147	0·05173	0·04323	0·03574	0·02917	64
65	0·05858	0·07520	0·04128	0·03412	0·02785	0·02243	65
66	0·04727	0·06126	0·03258	0·02660	0·02142	0·01698	66
67	0·03773	0·04944	0·02539	0·02045	0·01621	0·01266	67
68	0·02978	0·03946	0·01952	0·01548	0·01209	0·00931	68
69	0·02321	0·03114	0·01478	0·01154	0·00889	0·00678	69
70	0·01785	0·02427	0·01101	0·00848	0·00647	0·00488	70
71	0·01351	0·01867	0·00809	0·00618	0·00466	0·00345	71
72	0·01007	0·01413	0·00590	0·00445	0·00330	0·00244	72
73	0·00740	0·01053	0·00425	0·00315	0·00233	0·00174	73
74	0·00539	0·00774	0·00301	0·00222	0·00166	0·00126	74
75	0·00389	0·00564	0·00212	0·00159	0·00120	0·00091	75
76	0·00275	0·00406	0·00151	0·00115	0·00087	0·00066	76
77	0·00194	0·00287	0·00109	0·00083	0·00063	0·00048	77
78	0·00138	0·00203	0·00079	0·00060	0·00046	0·00034	78
79	0·00100	0·00145	0·00058	0·00044	0·00033	0·00024	79
80	0·00073	0·00105	0·00042	0·00031	0·00023	0·00016	80
81	0·00053	0·00076	0·00030	0·00022	0·00015	0·00010	81
82	0·00038	0·00055	0·00021	0·00015	0·00010	0·00006	82
83	0·00027	0·00040	0·00014	0·00009	0·00005	0·00002	83
84	0·00019	0·00029	0·00009	0·00005	0·00002		84
85	0·00013	0·00020	0·00005	0·00002			85
86	0·00008	0·00013	0·00002				86
87	0·00004	0·00008					87
88	0·00002	0·00004					88
89	0·00000	0·00001					89
	Age 89.	**Age 88.**	**Age 87.**	**Age 86.**	**Age 85.**	**Age 84.**	
0	2·57704	2·68338	2·77593	2·92831	3·11514	3·32856	0
1	1·82269	1·93321	2·02229	2·15279	3·32214	2·51970	1
2	1·28634	1·36732	1·45694	1·56833	1·70716	1·87828	2
3	0·91798	0·96497	1·03046	1·12988	1·24368	1·38084	3
4	0·66295	0·68863	0·72724	0·79914	0·89600	1·00596	4
5	0·48131	0·49732	0·51898	0·56399	0·63372	0·72473	5
6	0·35031	0·36106	0·37480	0·40248	0·44724	0·51259	6
7	0·25375	0·26279	0·27211	0·29066	0·31916	0·36175	7
8	0·18109	0·19035	0·19805	0·21103	0·23050	0·25816	8
9	0·12674	0·13584	0·14346	0·15359	0·16734	0·18644	9
10	0·08568	0·09508	0·10238	0·11126	0·12180	0·13536	10
11	0·05339	0·06428	0·07165	0·07940	0·08823	0·09852	11
12	0·02923	0·04005	0·04844	0·05557	0·06296	0·07136	12
13	0·01264	0·02193	0·03018	0·03757	0·04407	0·05093	13
14	0·00307	0·00948	0·01653	0·02341	0·02979	0·03564	14
15		0·00230	0·00715	0·01282	0·01856	0·02410	15
16			0·00173	0·00554	0·01016	0·01501	16
17				0·00135	0·00439	0·00822	17
18					0·00107	0·00355	18
19						0·00086	19

Years defd.	Age 20.	Age 21.	Age 22.	Age 23.	Age 24.	Age 25.	Years defd.
0	18·36170	18·23196	18·09386	17·95015	17·80058	17·64486	0
1	17·40695	17·27710	17·13905	16·99539	16·84586	16·69036	1
2	16·49530	16·36539	16·22742	16·08386	15·93458	15 77933	2
3	15·62484	15·49492	15·35708	15·21379	15·06480	14 91014	3
4	14·79376	14·66386	14·52633	14·38336	14·23497	14·08165	4
5	14·00031	13·87061	13·73342	13·59107	13·44400	13·29286	5
6	13·24296	13·11350	12·97693	12·83587	12·69093	12·54207	6
7	12·52010	12·39115	12·25586	12·11686	11·97413	11·82752	7
8	11·83044	11·70263	11·56934	11·43249	11·29194	11·14741	8
9	11·17308	11·04710	10·91590	10·78116	10·64263	10·50004	9
10	10·54721	10·42315	10·29400	10·16122	10·02457	9·88388	10
11	9·95150	9·82932	9·70207	9·57112	9·43632	9·29750	11
12	9·38454	9·26412	9·13863	9·00948	8·87649	8·73963	12
13	8·84491	8·72611	8·60237	8·47497	8·34388	8·20903	13
14	8·33125	8·21406	8·09202	7·96645	7·83731	7·70454	14
15	7·84236	7·72674	7·60647	7·48279	7·35566	7·22521	15
16	7·37710	7·26311	7·14467	7·02293	6·89803	6·77031	16
17	6·93445	6·82216	6·70559	6·58601	6·46374	6·33893	17
18	6·51345	6·40290	6·28841	6·17135	6·05184	5·93011	18
19	6·11316	6·00455	5·89249	5·77814	5·66158	5·54274	19
20	5·73284	5·62650	5·51705	5·40548	5·29175	5·17578	20
21	5·37190	5·26800	5·16123	5·05239	4·94141	4·82817	21
22	5·02962	4·92825	4·82409	4·71789	4·60954	4·49887	22
23	4·70524	4·60633	4·50471	4·40103	4·29515	4·18686	23
24	4·39789	4·30136	4·20216	4·10086	3·99727	3·89104	24
25	4·10672	4·01247	3·91556	3·81646	3·71484	3·61048	25
26	3·83091	3·73881	3·64401	3·54680	3·44699	3·34434	26
27	3·56963	3·47951	3·38654	3·29107	3·19290	3 09208	27
28	3·32207	3·23367	3·14236	3·04847	2·95207	2·85322	28
29	3·08734	3·00051	2·91072	2·81853	2·72402	2·62725	29
30	2·86474	2·77933	2·69117	2·60080	2·50829	2·41365	30
31	2·65356	2·56970	2·48328	2·39483	2·30435	2·21194	31
32	2·45341	2·37118	2·28661	2·20012	2·11178	2·02168	32
33	2·26389	2·18340	2·10070	2·01626	1·93013	1·84255	33
34	2·08459	2·00588	1·92515	1·84282	1·75912	1·67449	34
35	1·91511	1·83825	1·75955	1·67955	1·59866	1·51746	35
36	1·75506	1·68012	1·60365	1·52635	1·44874	1·37152	36
37	1·60410	1·53126	1·45738	1·38321	1·30941	1·23622	37
38	1·46197	1·39159	1·32071	1·25018	1·18024	1·11099	38
39	1·32862	1·26109	1·19369	1·12685	1·06068	0·99518	39
40	1·20402	1·13981	1·07593	1·01270	0·95011	0·88825	40
41	1·08823	1·02736	0·96694	0·90714	0·84803	0·78966	41
42	0·98088	0·92329	0·86615	0·80967	0·75391	0·69890	42
43	0·88151	0·82705	0·77308	0·71980	0·66725	0·61550	43
44	0·78962	0·73819	0·68728	0·63707	0·58762	0·53902	44
45	0·70478	0·65625	0·60828	0·56104	0·51461	0·46910	45
46	0·62656	0·58082	0·53569	0·49134	0·44786	0·40535	46
47	0·55454	0·51151	0·46913	0·42760	0·38699	0·34766	47
48	0·48836	0·44796	0·40828	0·36949	0·33191	0·29595	48
49	0·42769	0·38985	0·35279	0·31689	0·28255	0·25013	49
50	0·37221	0·33687	0·30258	0·26977	0·23880	0·21004	50
51	0·32162	0·28892	0·25758	0·22800	0·20052	0·17517	51
52	0·27584	0·24595	0·21770	0·19145	0·16724	0·14510	52
53	0·23482	0·20787	0·18280	0·15967	0·13853	0·11929	53
54	0·19846	0·17455	0·15246	0·13226	0·11388	0·09717	54

Years defer'd	Age 20.	Age 21.	Age 22.	Age 23.	Age 24.	Age 25.	Years defer'd
55	0·16665	0·14558	0·12630	0·10873	0·09277	0·07842	55
56	0·13899	0·12058	0·10384	0·08857	0·07487	0·06259	56
57	0·11513	0·09913	0·08457	0·07148	0·05975	0·04940	57
58	0·09465	0·08075	0·06825	0·05705	0·04716	0·03851	58
59	0·07710	0·06517	0·05447	0·04503	0·03676	0·02961	59
60	0·06222	0·05201	0·04300	0·03510	0·02827	0·02241	60
61	0·04966	0·04106	0·03351	0·02700	0·02140	0·01671	61
62	0·03920	0·03200	0·02577	0·02043	0·01595	0·01228	62
63	0·03055	0·02461	0·01950	0·01523	0·01173	0·00895	63
64	0·02349	0·01863	0·01454	0·01120	0·00854	0·00645	64
65	0·01778	0·01389	0·01069	0·00816	0·00615	0·00456	65
66	0·01326	0·01021	0·00779	0·00588	0·00435	0·00322	66
67	0·00975	0·00744	0·00561	0·00416	0·00307	0·00230	67
68	0·00710	0·00536	0·00397	0·00293	0·00219	0·00166	68
69	0·00512	0·00380	0·00280	0·00209	0·00158	0·00120	69
70	0·00362	0·00267	0·00200	0·00151	0·00115	0·00088	70
71	0·00255	0·00191	0·00144	0·00110	0·00084	0·00063	71
72	0·00182	0·00138	0·00105	0·00080	0·00061	0·00045	72
73	0·00132	0·00100	0·00076	0·00058	0·00043	0·00032	73
74	0·00096	0·00073	0·00055	0·00041	0·00030	0·00021	74
75	0·00070	0·00053	0·00039	0·00029	0·00020	0·00013	75
76	0·00050	0·00038	0·00028	0·00020	0·00013	0·00007	76
77	0·00036	0·00026	0·00019	0·00012	0·00007	0·00003	77
78	0·00025	0·00018	0·00012	0·00007	0·00003	0·00001	78
79	0·00017	0·00011	0·00006	0·00003	0·00001		79
80	0·00011	0·00006	0·00003	0·00001			80
81	0·00006	0·00003	0·00001				81
82	0·00003	0·00001					82
83	0·00001						83
	Age 83.	Age 82.	Age 81.	Age 80.	Age 79.	Age 78.	
0	3·53410	3·74634	3·95310	4·18289	4·39345	4·62166	0
1	2·71763	2·92008	3·12023	3·33839	3·54577	3·76476	1
2	2·05723	2·24547	3·43205	2·63503	2·82990	2·03838	2
3	1·53354	1·69981	1·87019	2·05387	2·23367	2·42495	3
4	1·12740	1·26710	1·41573	1·57938	1·74103	1·91404	4
5	0·82132	0·93153	1·05534	1·19558	1·33881	1·49189	5
6	0·51971	0·67863	0·77585	0·89123	1·01347	1·14723	6
7	0·41851	0·48891	0·56521	0·65520	0·75548	0·86845	7
8	0·29536	0·34580	0·40720	0·47732	0·55540	0·64737	8
9	0·21078	0·24404	0·28800	0·34388	0·40662	0·47593	9
10	0·15222	0·17416	0·20326	0·24322	0·29150	0·34672	10
11	0·11051	0·12577	0·14505	0·17165	0·20617	0·24979	11
12	0·08044	0·09131	0·10475	0·12249	0·14550	0·17667	12
13	0·05826	0·06646	0·07605	0·08846	0·10384	0·12468	13
14	0·04158	0·04814	0·05535	0·06423	0·07499	0·08898	14
15	0·02910	0·03435	0·04010	0·04675	0·05444	0·06426	15
16	0·01967	0·02405	0·02861	0·03386	0·03963	0·04665	16
17	0·01226	0·01626	0·02003	0·02416	0·02870	0·03396	17
18	0·00671	0·01013	0·01354	0·01691	0·02048	0·02460	18
19	0·00290	0·00555	0·00844	0·01143	0·01434	0·01755	19
20	0·00070	0·00240	0·00462	0·00712	0·00969	0·01228	20
21		0·00058	0·00200	0·00390	0·00604	0·00381	21
22			0·00048	0·00169	0·00331	0·00517	22
23				0·00041	0·00143	0·00283	23
24					0·00035	0·00123	24
25						0·00030	25

Years defer'd	Age 26.	Age 27.	Age 28.	Age 29.	Age 30.	Age 31.	Years defer'd
0	17·48586	17·32028	17·15412	16·99683	16·85215	16·70511	0
1	16·53141	16·36621	16·20094	16·04474	15·90033	15·75339	1
2	15·62080	15·45682	15·29344	15·13852	14·99445	14·84754	2
3	14·75282	14·59100	14·42965	14·27605	14·13224	13·98528	3
4	13·92643	13·76688	13·60756	13·45515	13·31153	13·16461	4
5	13·13985	12·98256	12·82510	12·67375	12·53039	12·38359	5
6	12·39125	12·23603	12·08230	11·93004	11·78700	11·64054	6
7	11·67873	11·52544	11·37141	11·22227	11·07975	10·93382	7
8	11·00050	10·84911	10·69678	10·54890	10·40708	10·26188	8
9	10·35498	10·20547	10·05494	9·90846	9·76750	9·62345	9
10	9·74065	9·59311	9·44449	9·29953	9·15983	9·01756	10
11	9·15618	9·01070	8·86407	8·72097	8·58313	8·44300	11
12	8·60029	8·45694	8·31260	8·17190	8·03625	7·89847	12
13	8·07176	7·93080	7·78924	7·65122	7·51796	7·38252	13
14	7·56958	7·43148	7·29294	7·15776	7·02687	6·89376	14
15	7·09300	6·95797	6·82259	6·69020	6·56165	6·43076	15
16	6·64106	6·50923	6·37693	6·24727	6·12096	5·99216	16
17	6·21275	5·08403	5·95474	5·82769	5·70349	5·57659	17
18	5·80692	5·68124	5·55481	5·43022	5·30794	5·18258	18
19	5·42248	5·29967	5·17595	5·05363	4·93290	4·80890	19
20	5·05829	4·93822	4·81699	4·69656	4·57722	4·45441	20
21	4·71330	4·59574	4·47664	4·35792	4·23981	4·11843	21
22	4·38642	4·27102	4·15386	4·03668	3·92002	3·80028	22
23	4·07649	3·96307	3·84766	3·73221	3·61720	3·49931	23
24	3·78257	3·67093	3·55744	3·44389	3·33073	3·21480	24
25	3·50374	3·39405	3·28263	3·17115	3·05993	2·94614	25
26	3·23946	3·13186	3·02266	2·91332	2·80421	2·69273	26
27	2·98921	2·88382	2·77690	2·66986	2·56300	2·45415	27
28	2·75248	2·64936	2·54484	2·44020	2·33592	2·23029	28
29	2·52869	2·42795	2·32594	2·22400	2·12285	2·02114	29
30	2·31737	2·21911	2·11986	2·02114	1·92377	1·82676	30
31	2·11804	2·02249	1·92650	1·83160	1·73877	1·64655	31
32	1·93037	1·83801	1·74583	1·65545	1·56723	1·47975	32
33	1·75430	1·66565	1·57793	1·49214	1·40846	1·32550	33
34	1·58978	1·50546	1·42227	1·34098	1·26165	1·18309	34
35	1·43689	1·35694	1·27819	1·20120	1·12609	1·05177	35
36	1·29514	1·21948	1·14495	1·07214	1·00110	0·93088	36
37	1·16394	1·09236	1·02193	0·95314	0·88603	0·81979	37
38	1·04261	0·97500	0·90851	0·84358	0·78030	0·71794	38
39	0·93059	0·86678	0·80408	0·74291	0·68335	0·62481	39
40	0·82730	0·76715	0·70812	0·65061	0·59471	0·53989	40
41	0·73221	0·67560	0·62015	0·56622	0·51388	0·46305	41
42	0·64483	0·59166	0·53970	0·48926	0·44074	0·39419	42
43	0·56471	0·51492	0·46635	0·41962	0·37520	0·33315	43
44	0·49146	0·44493	0·39997	0·35722	0·31710	0·27975	44
45	0·42467	0·38160	0·34049	0·30191	0·26628	0·23331	45
46	0·36422	0·32485	0·28777	0·25352	0·22207	0·19326	46
47	0·31006	0·27455	0·24165	0·21143	0·18395	0·15888	47
48	0·26205	0·23055	0·20153	0·17513	0·15123	0·12942	48
49	0·22005	0·19228	0·16693	0·14398	0·12319	0·10445	49
50	0·18352	0·15927	0·13724	0·11728	0·09942	0·08336	50
51	0·15201	0·13093	0·11179	0·09466	0·07935	0·06580	51
52	0·12497	0·10666	0·09022	0·07555	0·06263	0·05129	52
53	0·10180	0·08608	0·07201	0·05963	0·04882	0·03944	53
54	0·08216	0·06870	0·05684	0·04648	0·03754	0·02985	54

Years defer'd	Age 26.	Age 27.	Age 28.	Age 29.	Age 30.	Age 31.	Years defer'd
55	0·06557	0·05423	0·04430	0·03574	0·02842	0·02225	55
56	0·05176	0·04227	0·03407	0·02705	0·02118	0·01636	56
57	0·04034	0·03250	0·02579	0·02017	0·01557	0·01192	57
58	0·03102	0·02460	0·01922	0·01483	0·01134	0·00859	58
59	0·02348	0·01834	0·01413	0·01080	0·00817	0·00607	59
60	0·01750	0·01348	0·01030	0·00778	0·00578	0·00429	60
61	0·01287	0·00982	0·00742	0·00550	0·00408	0·00306	61
62	0·00938	0·00708	0·00525	0·00388	0·00291	0·00221	62
63	0·00675	0·00500	0·00370	0·00277	0·00210	0·00160	63
64	0·00478	0·00353	0·00264	0·00200	0·00153	0·00117	64
65	0·00337	0·00252	0·00191	0·00145	0·00111	0·00085	65
66	0·00241	0·00182	0·00138	0·00106	0·00080	0·00060	66
67	0·00174	0·00132	0·00101	0·00077	0·00057	0·00042	67
68	0·00126	0·00096	0·00073	0·00055	0·00040	0·00029	68
69	0·00092	0·00070	0·00052	0·00038	0·00027	0·00018	69
70	0·00066	0·00050	0·00036	0·00026	0·00017	0·00010	70
71	0·00047	0·00035	0·00025	0·00016	0·00009	0·00004	71
72	0·00033	0·00023	0·00015	0·00009	0·00004	0·00001	72
73	0·00022	0·00015	0·00008	0·00004	0·00001		73
74	0·00014	0·00008	0·00004	0·00001			74
75	0·00008	0·00003	0·00001				75
76	0·00003	0·00001					76
77	0·00001						77
	Age 77.	**Age 76.**	**Age 75.**	**Age 74.**	**Age 73.**	**Age 72.**	
0	4·82473	5·02400	5·23901	5·45812	5·72465	6·02548	0
1	3·96649	4·16147	4·36932	4·58328	4·83822	5·12945	1
2	3·23107	3·42121	3·61918	3·82245	4·06274	4·33519	2
3	2·60765	2·78688	2·97539	3·16620	3·38831	3·64034	3
4	2·08118	2·24918	2·42372	2·60299	2·80660	3·03603	4
5	1·64270	1·79508	1·95608	2·12037	2·30735	2·51480	5
6	1·28040	1·41688	1·56116	1·71126	1·87955	2·06746	6
7	0·98460	1·10438	1·23224	1·36577	1·51691	1·68413	7
8	0·74534	0·84924	0·96047	1·07801	1·21065	1·35919	8
9	0·55560	0·64287	0·73858	0·84026	0·95558	1·08478	9
10	0·40846	0·47922	0·55910	0·64614	0·74482	0·85623	10
11	0·29757	0·35231	0·41677	0·48912	0·57275	0·66739	11
12	0·21438	0·25666	0·30640	0·36461	0·43357	0·51320	12
13	0·15162	0·18491	0·22321	0·26805	0·32320	0·38849	13
14	0·10701	0·13078	0·16081	0·19528	0·23761	0·28960	14
15	0·07636	0·09230	0·11374	0·14068	0·17310	0·21290	15
16	0·05515	0·06587	0·08027	0·09950	0·12471	0·15510	16
17	0·04004	0·04757	0·05728	0·07022	0·08820	0·11174	17
18	0·02914	0·03453	0·04137	0·05011	0·06225	0·07903	18
19	0·02111	0·02514	0·03003	0·03619	0·04442	0·05578	19
20	0·01506	0·01821	0·02186	0·02628	0·03208	0·03980	20
21	0·01054	0·01299	0·01583	0·01912	0·02329	0·02875	21
22	0·00713	0·00909	0·01130	0·01385	0·01695	0·02087	22
23	0·00444	0·00615	0·00791	0·00989	0·01228	0·01519	23
24	0·00243	0·00383	0·00535	0·00692	0·00876	0·01100	24
25	0·00105	0·00210	0·00333	0·00468	0·00613	0·00785	25
26	0·00026	0·00091	0·00182	0·00291	0·00415	0·00550	26
27		0·00022	0·00079	0·00160	0·00258	0·00372	27
28			0·00019	0·00069	0·00141	0·00231	28
29				0·00017	0·00061	0·00127	29
30					0·00015	0·00055	30
31						0·00013	31

Years defer'd	Age 32.	Age 33.	Age 34.	Age 35.	Age 36.	Age 37.	Years defer'd
0	16·55245	16·39072	16·21943	16·04123	15·85577	15·66586	0
1	15·60065	15·43885	15·26765	15·08955	14·90438	14·71476	1
2	14·69466	14·53288	14·36187	14·18413	13·99951	13·81045	2
3	13·83236	13·67069	13·50011	13·32299	13·13916	12·95125	3
4	13·01173	12·85041	12·68050	12·50422	12·32172	12·13584	4
5	12·23099	12·07024	11·90121	11·72628	11·54595	11·36260	5
6	11·48842	11·32845	11·16079	10·98800	10·81029	10·62978	6
7	10·78239	10·62367	10·45811	10·28789	10·11309	9·93542	7
8	10·11158	9·95480	9·79676	9·62438	9·45248	9·27764	8
9	9·47496	9·32052	9·16025	8·99569	8·82668	8·65453	9
10	8·87125	8·71940	8·56188	8·40013	8·23386	8·06427	10
11	8·29911	8·14983	7·99504	7·83596	7·67228	7·50499	11
12	7·75699	7·61027	7·45808	7·30152	7·14019	6·97472	12
13	7·24344	7·09915	6·94941	6·79515	6·63569	6·47182	13
14	6·75695	6·61496	6·46745	6·31503	6·15724	5·99475	14
15	6·29611	6·15620	6·01049	5·85970	5·70336	5·54259	15
16	5·85946	5·72123	5·57712	5·42775	5·27318	5·11443	16
17	5·44545	5·30871	5·16600	5·01835	4·86583	4·70938	17
18	5·05282	4·91738	4·77635	4·63069	4·48047	4·32649	18
19	4·68035	4·54648	4·40738	4·26395	4·11619	3·96493	19
20	4·32733	4·19527	4·05832	3·91728	3·77220	3·62387	20
21	3·99304	3·86301	3·72837	3·58991	3·44773	3·30279	21
22	3·67681	3·54894	3·41679	3·28112	3·14225	3·00153	22
23	3·37787	3·25235	3·12289	2·99040	2·85564	2·72005	23
24	3·09558	2·97260	2·84619	2·71764	2·58784	2·45846	24
25	2·82931	2·70922	2·58658	2·46278	2·33896	2·21593	25
26	2·57863	2·46210	2·34402	2·22593	2·10822	1·99145	26
27	2·34342	2·23121	2·11859	2·00634	1·89465	1·78387	27
28	2·12366	2·01663	1·90959	1·80310	1·69716	1·59220	28
29	1·91942	1·81769	1·71614	1·61514	1·51481	1·41548	29
30	1·73007	1·63355	1·53725	1·44161	1·34668	1·25278	30
31	1·55481	1·46327	1·37208	1·28160	1·19188	1·10328	31
32	1·39274	1·30605	1·21979	1·13428	1·04965	0·96620	32
33	1·24310	1·16109	1·07958	0·99892	0·91924	0·84087	33
34	1·10512	1·02763	0·95075	0·87482	0·80000	0·72659	34
35	0·97810	0·90500	0·83263	0·76134	0·69127	0·62317	35
36	0·86137	0·79256	0·72463	0·65787	0·59288	0·53050	36
37	0·75436	0·68975	0·62614	0·56423	0·50471	0·44835	37
38	0·65651	0·59601	0·53702	0·48032	0·42656	0·37649	38
39	0·56728	0·51117	0·45716	0·40595	0·35819	0·31399	39
40	0·48653	0·43516	0·38637	0·34088	0·29873	0·26009	40
41	0·41418	0·36778	0·32444	0·28429	0·24744	0·21382	41
42	0·35005	0·30883	0·27058	0·23549	0·20343	0·17418	42
43	0·29394	0·25756	0·22413	0·19360	0·16571	0·14057	43
44	0·24515	0·21334	0·18426	0·15770	0·13374	0·11219	44
45	0·20306	0·17539	0·15010	0·12727	0·10674	0·08855	45
46	0·16694	0·14287	0·12114	0·10158	0·08425	0·06902	46
47	0·13599	0·11531	0·09668	0·08018	0·06567	0·05308	47
48	0·10975	0·09203	0·07631	0·06249	0·05050	0·04018	48
49	0·08759	0·07264	0·05948	0·04806	0·03823	0·02995	49
50	0·06914	0·05662	0·04574	0·03638	0·02849	0·02202	50
51	0·05389	0·04354	0·03462	0·02712	0·02095	0·01604	51
52	0·04144	0·03296	0·02581	0·01994	0·01526	0·01156	52
53	0·03137	0·02457	0·01897	0·01452	0·01099	0·00817	53
54	0·02338	0·01806	0·01382	0·01046	0·00778	0·00577	54

Years defer'd	**Age 32.**	**Age 33.**	**Age 34.**	**Age 35.**	**Age 36.**	**Age 37.**	Years defer'd
55	0·01719	0·01316	0·00996	0·00740	0·00549	0·00412	55
56	0·01252	0·00948	0·00704	0·00522	0·00392	0·00297	56
57	0·00902	0·00670	0·00497	0·00373	0·00283	0·00216	57
58	0·00638	0·00473	0·00355	0·00269	0·00205	0·00157	58
59	0·00450	0·00338	0·00256	0·00195	0·00149	0·00114	59
60	0·00321	0·00244	0·00186	0·00142	0.00108	0·00081	60
61	0·00232	0·00177	0·00135	0·00103	0·00077	0·00057	61
62	0·00169	0·00129	0·00098	0·00073	0·00054	0·00038	62
63	0·00123	0·00093	0·00070	0·00051	0·00037	0·00024	63
64	0·00089	0·00067	0·00049	0·00035	0·00023	0·00013	64
65	0·00063	0·00047	0·00033	0·00022	0·00012	0·00006	65
66	0·00044	0·00032	0·00021	0·00012	0·00005	0·00001	66
67	0·00030	0·00020	0·00011	0·00005	0·00001		67
68	0·00019	0·00011	0·00005	0·00001			68
69	0·00010	0·00005	0·00001				69
70	0·00004	0·00001					70
71	0·00001						71
	Age 71.	**Age 70.**	**Age 69.**	**Age 68.**	**Age 67.**	**Age 66.**	
0	6·35773	6·70936	7·04881	7·37975	7·69980	8·00966	0
1	5·45278	5·79748	6·13450	6·46288	6·78094	7·08899	1
2	4·64191	4·97228	5·30075	5·65456	5·93846	6·24302	2
3	3·92314	4·23287	5·54624	4·86012	5·16817	5·46737	3
4	3·29433	3·57743	3·87019	4·16834	4·46576	4·75819	4
5	2·74746	3·00404	3·27091	3·54848	3.83011	4·11150	5
6	2·27577	2·50536	2·74664	2·99902	3·26054	3·52627	6
7	1·87095	2·07523	2·29069	2·51833	2·75567	3·00189	7
8	1·52406	1·70608	1·89742	2·10028	2·31399	2·53707	8
9	1·23000	1·38976	1·55990	1·73970	1·92986	2·13042	9
10	0·98167	1·12161	1·27068	1·43024	1·59854	1·77676	10
11	0·77485	0·89517	1·02551	1·16506	1·31418	1·47173	11
12	0·60395	0·70657	0·81847	0·94027	1·07052	1·20993	12
13	0·46442	0·55073	0·64603	0·75043	0·86397	0·98560	13
14	0·35157	0·42350	0·50354	0·59232	0·68954	0·79543	14
15	0·26207	0·32059	0·38721	0·46169	0·54426	0·63484	15
16	0·19267	0·23898	0·29312	0·35503	0·42422	0·50109	16
17	0·14036	0·17569	0·21850	0·26875	0·32622	0·39057	17
18	0·10112	0·12799	0·16063	0·20034	0·24695	0·30034	18
19	0·07152	0·09221	0·11702	0·14728	0·18408	0·22736	19
20	0·05047	0·06522	0·08431	0·10730	0·13533	0·16948	20
21	0·03602	0·04603	0·05963	0·07730	0·09859	0·12460	21
22	0·02601	0·03285	0·04208	0·05467	0·07103	0·09077	22
23	0·01889	0·02372	0·03003	0·03858	0·05024	0·06539	23
24	0·01375	0·01722	0·02169	0·02753	0·03545	0·04625	24
25	0·00996	0·01253	0·01575	0·01989	0·02530	0·03264	25
26	0·00711	0·00908	0·01146	0·01444	0·01827	0·02329	26
27	0·00497	0·00648	0·00830	0·01051	0·01327	0·01682	27
28	0·00336	0·00453	0·00592	0·00761	0·00966	0·01221	28
29	0·00209	0·00307	0·00415	0·00543	0·00699	0·00889	29
30	0·00115	0·00191	0·00280	0·00380	0·00499	0·00644	30
31	0·00050	0·00105	0·00175	0·00257	0·00349	0·00460	31
32	0·00012	0·00045	0·00096	0·00160	0·00236	0·00322	32
33		0·00011	0·00041	0·00088	0·00147	0·00217	33
34			0·00010	0·00038	0·00081	0·00135	34
35				0·00009	0·00035	0·00074	35
36					0·00008	0·00032	36
37						0·00008	37

Years defer'd	Age 38.	Age 39.	Age 40.	Age 41.	Age 42.	Age 43.	Years defer'd
0	15·47130	15·27185	15·07363	14·88313	14·69465	14·50529	0
1	14·52049	14·32173	14·12459	13·93484	13·74694	13·55777	1
2	13·61712	13·42004	13·22463	13·03612	12·84896	12·66018	2
3	12·75979	12·56496	12·37172	12·18458	11·99829	11·80989	3
4	11·94679	11·75460	11·56357	11·37790	11·19246	11·00442	4
5	11·17629	10·98676	10·79801	10·61373	10·42910	10·24124	5
6	10·44623	10·25938	10·07279	9·88984	9·70581	9·51764	6
7	9·75464	9·57034	9·38579	9·20396	9·02004	8·83138	7
8	9·09949	8·91761	8·73486	8·55364	8·36967	8·18038	8
9	8·47888	8·29916	8·11770	7·93690	7·75270	7·56336	9
10	7·89085	7·71277	7·53238	7·35183	7·16794	6·97910	10
11	7·33331	7·15666	6·97714	6·79731	6·61422	6·42637	11
12	6·80456	6·62911	6·45088	6·27222	6·09039	5·90389	12
13	6·30296	6·12910	5·95255	5·77548	5·59522	5·41050	13
14	5·82755	5·65562	5·48112	5·30591	5·12763	4·94510	14
15	5·37738	5·20772	5·03550	4·86250	4·68657	4·50696	15
16	4·95150	4·78431	4·61468	4·44424	4·27133	4·09587	16
17	4·54893	4·38449	4·21773	4·05047	3·88173	3·71176	17
18	4·16878	4·00735	3·84404	3·68102	3·51770	3·35479	18
19	3·81019	3·65229	3·49341	3·33581	3·17940	3·02384	19
20	3·47262	3·31915	3·16580	3·01500	2·86575	2·71752	20
21	3·15585	3·00788	2·86134	2·71757	2·57544	2·43425	21
22	2·85990	2·71861	2·57907	2·44228	2·30698	2·17270	22
23	2·58486	2·45042	2·31780	2·18769	2·05911	1·93155	23
24	2·32986	2·20219	2·07620	1·95264	1·83057	1·70953	24
25	2·09384	1·97263	1·85312	1·73591	1·62015	1·50552	25
26	1·87558	1·76068	1·64744	1·53638	1·42681	1·31847	26
27	1·67406	1·56526	1·45807	1·35303	1·24954	1·14745	27
28	1·48825	1·38534	1·28407	1·18493	1·08746	0·99150	28
29	1·31719	1·22002	1·12454	1·03123	0·93966	0·85037	29
30	1·16000	1·06845	0·97867	0·89107	0·80591	0·72391	30
31	1·01588	0·92985	0·84566	0·76424	0·68607	0·61182	31
32	0·88411	0·80348	0·72529	0·65059	0·57983	0·51376	32
33	0·76395	0·68911	0·61743	0·54985	0·48690	0·42847	33
34	0·65521	0·58664	0·52183	0·46172	0·40607	0·35491	34
35	0·55777	0·49580	0·43819	0·38507	0·33636	0·29178	35
36	0·47141	0·41633	0·36545	0·31896	0·27652	0·23768	36
37	0·39585	0·34722	0·30271	0·26223	0·22525	0·19182	37
38	0·33014	0·28761	0·24886	0·21361	0·18179	0·15309	38
39	0·27346	0·23645	0·20272	0·17239	0·14509	0·12084	39
40	0·22481	0·19261	0·16361	0·13759	0·11452	0·09419	40
41	0·18313					0·07243	41
42	0·14780	0·15545	0·13057	0·10860	0·08926	0·05483	42
43	0·11796	0·12406	0·10306	0·18465	0·06864	0·04087	43
44	0·09311	0·09792	0·08033	0·06509	0·05196	0·03005	44
45	0·07257	0·07633	0·06177	0·04927	0·03873	0·02189	45
46	0·05581	0·05869	0·04676	0·03673	0·02848	0·01577	46
47	0·04224	0·04443	0·03486	0·02700	0·02074	0·01115	47
48	0·03149	0·03312	0·02563	0·01967	0·01495	0·00787	48
49	0·02315	0·02435	0·01867	0·01417	0·01057	0·00562	49
50	0·01687	0·01774	0·01345	0·01002	0·00746	0·00406	50
51	0·01215	0·01278	0·00951	0·00707	0·00532	0·00295	51
52	0·00859	0·00904	0·00671	0·00505	0·00384	0·00214	52
53	0·00606	0·00638	0·00479	0·00365	0·00279	0·00155	53
54	0·00433	0·00455	0·00346	0·00265	0·00203	0·00111	54

Years defer'd	Age 38.	Age 39.	Age 40.	Age 41.	Age 42.	Age 43.	Years defer'd
55	0·00313	0·00329	0·00251	0·00193	0·00147	0·00078	55
56	0·00227	0·00239	0·00183	0·00140	0·00105	0·00052	56
57	0·00165	0·00174	0·00132	0·00100	0·00073	0·00033	57
58	0·00120	0·00126	0·00095	0·00070	0·00050	0·00018	58
59	0·00085	0·00090	0·00066	0·00047	0.00031	0·00008	59
60	0·00060	0·00063	0·00045	0·00029	0·00017	0·00002	60
61	0·00040	0·00042	0·00028	0·00016	0·00007		61
62	0·00025	0·00026	0·00015	0·00007	0·00002		62
63	0·00014	0·00014	0·00007	0·00002			63
64	0·00006	0·00006	0·00002				64
65	0·00001	0·00002					65
	Age 65.	**Age 64.**	**Age 63.**	**Age 62.**	**Age 61.**	**Age 60.**	
0	8·30719	8·59330	8·87150	9·13676	9·39809	9.66334	0
1	7·38516	7·67001	7·94674	8·21119	8·47096	8·73400	1
2	6·53627	6·81870	7·09291	7·35526	7·61284	7·87238	2
3	5·75626	6·03492	6·30566	6·56498	6·81928	7·07490	3
4	5·04109	5·31474	6·58085	5·83632	6·08659	6·33741	4
5	4·38720	4·65443	4·91486	5·16547	6·41103	5·65650	5
6	3·79093	4·05069	4·30423	4·54904	4·78906	5·02867	6
7	3·25133	3·50016	3·74592	3·98386	4·21755	4·45065	7
8	2·76784	3·00195	3·23680	3·46711	3·69356	3·91953	8
9	2·33926	2·55554	2·77608	2·99589	3·21446	3·43256	9
10	1·96432	2·15983	2·36326	2·56946	2·77757	2·98731	10
11	1·63823	1·81365	1·99732	2·18736	2·38222	2·58130	11
12	1·35698	1·51258	1·67719	1·84866	2·02797	2·21388	12
13	1·11559	1·25289	1·39877	1·55235	1·71395	1·88466	13
14	0·90875	1·03003	1·15863	1·29466	1·43923	1·59284	14
15	0·73342	0·83905	0·95253	1·07240	1·20032	1·33753	15
16	0·58534	0·67716	0·77592	0·88163	0·99424	1·11550	16
17	0·46202	0·54045	0·62621	0·71817	0·81738	0·92399	17
18	0·36012	0·42658	0·49978	0·57960	0·66583	0·75963	18
19	0·27693	0·33250	0·39448	0·46258	0·53737	0·61878	19
20	0·20963	0·25568	0·30748	0·36512	0·42887	0·49939	20
21	0·15627	0·19355	0·23644	0·28459	0·33852	0·39857	21
22	0·11488	0·14428	0·17899	0·21885	0·26386	0·31460	22
23	0·08369	0·10607	0·13342	0·16567	0·20290	0·24521	23
24	0·06030	0·07727	0·09809	0·12349	0·15359	0·18856	24
25	0·04265	0·05567	0·07146	0·09079	0·11449	0·14274	25
26	0·03010	0·03937	0·05148	0·06614	0·08417	0·10640	26
27	0·02148	0·02779	0·03641	0·04765	0·06132	0·07822	27
28	0·01551	0·01983	0·02570	0·03370	0·04418	0·05699	28
29	0·01126	0·01432	0·01834	0·02378	0·03125	0·04106	29
30	0·00820	0·01040	0·01324	0·01697	0·02205	0·02904	30
31	0·00594	0·00757	0·00961	0·01226	0·01574	0·02049	31
32	0·00424	0·00548	0·00700	0·00890	0·01136	0·01462	32
33	0·00297	0·00391	0·00506	0·00648	0·00825	0·01056	33
34	0·00200	0·00274	0·00362	0·00469	0·00601	0·00767	34
35	0·00125	0·00185	0·00253	0·00335	0·00435	0·00558	35
36	0·00068	0·00115	0·00171	0·00234	0·00310	0·00404	36
37	0·00030	0·00063	0·00107	0·00158	0·00217	0·00289	37
38	0·00007	0·00027	0·00058	0·00099	0·00147	0·00202	38
39		0·00007	0·00025	0·00055	0·00092	0·00137	39
40			0·00006	0·00023	0·00050	0·00085	40
				0·00006	0·00022	0·00047	
41					0·00005	0·00020	41
42						0·00005	42

TABLE XXIV.—DEFERRED ANNUITIES, CARLISLE 4 PER CENT.

Years defer'd	Age 44.	Age 45.	Age 46.	Age 47.	Age 48.	Age 49.	Years defer'd
0	14·30874	14·10460	13·88927	13·66208	13·41913	13·15312	0
1	13·36143	13·15730	12·94198	12·71458	12·47099	12·20474	1
2	12·46404	12·25993	12·04443	11·81623	11·57180	11·30507	2
3	11·61395	11·40968	11·19342	10·96424	10·71878	10·45237	3
4	10·80850	10·60352	10·38634	10·15601	9·91030	9·64493	4
5	10·04482	9·83897	9·62071	9·38998	9·14474	8·88108	5
6	9·32055	9·11369	8·89505	8·66461	8·42050	8·15902	6
7	8·63349	8·42628	8·20791	7·97840	7·73588	7·47717	7
8	7·98230	7·77535	7·55787	7·32972	7·08940	6·83401	8
9	7·36567	7·15957	6·94339	6·71718	6·47959	6·22850	9
10	6·78233	6·57747	6·36313	6·13940	5·90549	5·66038	10
11	6·23090	6·02779	5·81580	5·59543	5·36683	5·12955	11
12	5·71019	5·50930	5·30051	5·08505	4·86353	4·63624	12
13	5·21901	5·02117	4·81703	4·60818	4·39580	4·17887	13
14	4·75660	4·56317	4·36529	4·16500	3·96215	3·75554	14
15	4·32274	4·13524	3·94548	3·75412	3·56078	3·36407	15
16	3·91735	3·73755	3·55625	3·37382	3·18960	3·00262	16
17	3·54061	3·36883	3·19600	3·02214	2·84690	2·66935	17
18	3·19133	3·02757	2·86285	2·69743	2·53092	2·36252	18
19	2·86804	2·71197	2·55525	2·39803	2·24000	2·08059	19
20	2·56908	2·42059	2·27164	2·12239	1·97269	1·82209	20
21	2·29305	2·15192	2·01053	1·86912	1·72760	1·58575	21
22	2·03854	1·90457	1·77060	1·63689	1·50351	1·37022	22
23	1·80422	1·67729	1·55062	1·42457	1·29916	1·17519	23
24	1·58891	1·46890	1·34948	1·23095	1·11424	1·00043	24
25	1·39150	1·27836	1·16607	1·05574	0·94855	0·84552	25
26	1·21101	1·10462	1·00009	0·89874	0·80167	0·71000	26
27	1·04642	0·94739	0·85137	0·75958	0·67318	0·59214	27
28	0·89747	0·80651	0·71954	0·63783	0·56143	0·49048	28
29	0·76401	0·68162	0·60421	0·53195	0·46504	0·40323	29
30	0·64571	0 57237	0·50391	0·44062	0·38232	0·32847	30
31	0·54221	0·47736	0·41740	0·36224	0·31143	0·26509	31
32	0·45220	0·39540	0·34315	0·29508	0·25134	0·21157	32
33	0·37457	0·32507	0·27953	0·23815	0·20060	0·16700	33
34	0·30794	0·26480	0·22559	0·19007	0·15833	0·13016	34
35	0·25084	0·21371	0·18005	0·15002	0·12341	0.10009	35
36	0·20245	0·17056	0·14211	0·11693	0·09490	0·07577	36
37	0·16157	0·13462	0·11077	0·08992	0·07184	0·05648	37
38	0·12753	0·10493	0·08518	0·06807	0·05355	0·04152	38
39	0·09940	0·08069	0·06448	0·05074	0·03937	0·03025	39
40	0·07644	0·06108	0·04807	0·03730	0·02868	0·02179	40
41	0·05786	0·04553	0·03534	0·02718	0·02066	0·01541	41
42	0·04313	0·03347	0·02574	0·01958	0·01461	0·01088	42
43	0·03171	0·02439	0·01855	0·01385	0·01031	0·00776	43
44	0·02310	0·01757	0·01312	0·00977	0·00736	0·00561	44
45	0·01664	0·01243	0.00926	0·00697	0·00532	0·00407	45
46	0·01177	0·00877	0·00661	0·00504	0·00386	0·00296	46
47	0·00831	0·00626	0·00477	0·00366	0·00281	0·00215	47
48	0·00593	0·00452	0·00346	0·00266	0·00203	0·00153	48
49	0·00428	0·00328	0·00252	0·00193	0·00145	0·00107	49
50	0·00311	0·00239	0·00183	0·00138	0·00102	0·00072	50
51	0·00226	0·00173	0·00130	0·00096	0·00069	0·00045	51
52	0·00164	0·00123	0·00091	0·00065	0·00043	0·00025	52
53	0·00117	0·00086	0·00062	0·00041	0·00023	0·00011	53
54	0·00082	0·00058	0·00038	0·00022	0·00010	0·00003	54

Years defer'd	Age 44.	Age 45.	Age 46.	Age 47.	Age 48.	Years defer'd
55	0·00055	0·00036	0·00021	0·00010	0·00002	55
56	0·00034	0·00020	0·00009	0·00002		56
57	0·00019	0·00009	0·00002			57
58	0·00008	0·00002				58
59	0·00002					59
	Age 59.	**Age 58.**	**Age 57.**	**Age 56.**	**Age 55.**	
0	9·96331	10·28647	10·62559	10·96606	11·29961	0
1	9·02896	9·34820	9·68415	10·02279	10·35531	1
2	8·16063	8·47154	8·80082	9·13476	9·46457	2
3	7·35557	7·65682	7·97549	8·30154	8·62600	3
4	6·61044	6·90146	7·20847	7·52303	7·83919	4
5	5·92137	6·20233	6·49735	6·79953	7·10404	5
6	5·28516	5·55581	5·83916	6·12875	6·42083	6
7	4·69855	4·95887	5·23049	5.50790	5·78741	7
8	4·15847	4·40847	4·66850	4·93376	5·20114	8
9	3·66222	3·90174	4·15034	4·40366	4·65897	9
10	3·20722	3·43613	3·67328	3·91489	4·15839	10
11	2·79120	3·00922	3·23492	3·46489	3·69685	11
12	2·41185	2·61888	2·83301	3·05140	3·27191	12
13	2·06855	2·26295	2·46553	2·67229	2·88146	13
14	1·76094	1·94084	2·13044	2·32566	2·52346	14
15	1·48827	1·65223	1·82720	2·00958	2·19613	15
16	1·24973	1·39639	1·55548	1·72354	1·89765	16
17	1·04227	1·17257	1·31462	1·46724	1·62754	17
18	0·86333	0·97792	1·10391	1·24004	1·38552	18
19	0·70976	0·81003	0·92066	1·04129	1·17098	19
20	0·57816	0·66594	0·76260	0·86843	0·98329	20
21	0·46661	0·54247	0·62695	0·71934	0·82006	21
22	0·37240	0·43780	0·51070	0·59138	0·67927	22
23	0·29394	0·34941	0·41217	0·48173	0·55844	23
24	0·22911	0·27580	0·32895	0·38878	0·45490	24
25	0·17618	0·21497	0·25965	0·31029	0·36713	25
26	0·13337	0·16531	0·20238	0·24492	0·29301	26
27	0·09942	0·12514	0·15563	0·19090	0·23128	27
28	0·07309	0·09328	0·11781	0·14680	0·18027	28
29	0·05325	0·06858	0·08782	0·11112	0·13862	29
30	0·03836	0·04996	0·06456	0·08284	0·10494	30
31	0·02713	0·03599	0·04703	0·06090	0·07822	31
32	0·01915	0·02546	0·03388	0·04437	0·05751	32
33	0·01366	0·01797	0·02397	0·03196	0·04189	33
34	0·00987	0·01282	0·01691	0·02261	0·03018	34
35	0·00716	0·00926	0·01207	0·01595	0·02135	35
36	0·00521	0·00672	0·00872	0·01139	0·01507	36
37	0·00378	0·00489	0·00633	0·00822	0·01075	37
38	0·00270	0·00354	0·00461	0·00597	0·00776	38
39	0·00189	0·00253	0·00334	0·00434	0·00564	39
40	0·00128	0·00177	0·00238	0·00315	0·00410	40
41	0·00079	0·00120	0·00167	0·00225	0·00297	41
42	0·00044	0·00075	0·00113	0·00157	0·00212	42
43	0·00019	0·00041	0·00070	0·00106	0·00148	43
44	0·00005	0·00018	0·00038	0·00066	0·00100	44
45		0·00004	0·00017	0·00036	0·00063	45
46			0·00004	0·00016	0·00034	46
47				0·00004	0·00015	47
48					0·00004	48
49						49

Years defer'd	Age 50.	Age 51.	Age 52.	Age 53.	Age 54.	Years defer'd
0	12·86902	12·56581	12·25793	11·94503	11·62673	0
1	11·92038	11·61802	11·31101	10·99902	10·68143	1
2	11·02127	10·72053	10·41521	10·10476	9·78879	2
3	10·16988	9·87149	9·56842	9·26031	8·94679	3
4	9·36446	9·06891	8·76879	8·46376	8·15409	4
5	8·60309	8·31102	8·01452	7·71368	7·41033	5
6	7·88414	7·59613	7·30442	7·01025	6·71539	6
7	7·20597	6·92310	6·63816	6·35284	6·06956	7
8	6·56751	6·29162	6·01564	5·74188	5·47079	8
9	5·96846	5·70160	5·43711	5·17543	4·91659	9
10	5·40874	5·15327	4·90073	4·65115	4·40409	10
11	4·88858	4·64489	4·40428	4·16632	3·93090	11
12	4·40631	4·17436	3·94518	3·71868	3·49460	12
13	3·95995	3·73923	3·52130	3·30593	3·09291	13
14	3·54716	3·33747	3·13046	2·92593	2·72382	14
15	3·16604	2·96704	2·77063	2·57676	2·38541	15
16	2·81464	2·62590	2·43999	2·25662	2·07599	16
17	2·49111	2·31262	2·13684	1·96391	1·79384	17
18	2·19383	2·02529	1·85967	1·69699	1·53851	18
19	1·92127	1·76259	1·60692	1·45544	1·30972	19
20	1·67205	1·52303	1·37819	1·23901	1·10692	20
21	1·44480	1·30624	1·17325	1·04716	0·92950	21
22	1·23915	1·11100	0·99158	0·87932	0·77520	22
23	1·05488	0·93981	0·83264	0·73335	0·64211	23
24	0·89154	0·78918	0·69442	0·60744	0·52789	24
25	0·74864	0·65817	0·57520	0·49939	0·43001	25
26	0·62436	0·54517	0·47288	0·40680	0·34705	26
27	0·51717	0·44820	0·38521	0·32831	0·27698	27
28	0·42518	0·36510	0·31088	0·26203	0·21862	28
29	0·34634	0·29465	0·24812	0·20682	0·17041	29
30	0·27952	0·23517	0·19584	0·16121	0·13104	30
31	0·22309	0·18562	0·15265	0·12396	0·09919	31
32	0·17608	0·14468	0·11738	0·09384	0·07394	32
33	0·13725	0·11126	0·08886	0·06995	0·05436	33
34	0·10554	0·08422	0·06624	0·05143	0·03960	34
35	0·07989	0·06278	0·04870	0·03746	0·02853	35
36	0·05956	0·04615	0·03548	0·02699	0·02018	36
37	0·04378	0·03362	0·02556	0·01909	0·01424	37
38	0.03190	0·02422	0·01808	0·01347	0·01016	38
39	0·02298	0·01713	0·01276	0·00961	0·00734	39
40	0·01626	0·01209	0·00910	0·00694	0·00533	40
41	0·01147	0·00863	0·00657	0·00504	0·00388	41
42	0·00819	0·00623	0·00477	0·00367	0·00281	42
43	0·00591	0:00452	0·00347	0.00266	0·00200	43
44	0·00429	0·00329	0·00252	0·00190	0·00140	44
45	0·00312	0·00239	0·00180	0·00133	0·00095	45
46	0·00226	0·00170	0·00126	0·00090	0·00059	46
47	0·00161	0·00119	0·00085	0·00056	0·00032	47
48	0·00113	0·00081	0·00053	0·00031	0·00014	48
49	0·00076	0·00050	0·00029	0·00013	0·00003	49
50	0·00048	0·00027	0·00013	0·00003		50
51	0·00026	0·00012	0·00003			51
52	0·00011	0·00003				52
53	0·00003					53

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